12

Energy and Power Spectral Densities

In this chapter we study energy and power spectra and their relations to signal duration, periodicity and correlation functions.

12.1 Energy Spectral Density

Let $f(t)$ be an electric potential in Volt applied across a resistance of $R = 1$ ohm. The total energy dissipated in such a resistance is given by

$$E = \int_{-\infty}^{\infty} \left\{ f^2(t) / R \right\} dt. \tag{12.1}$$

Since the resistance value is unity the dissipated energy may be also be referred to as *normalized energy*. In what follows we shall refer to it simply as the dissipated energy, with the implicit assumption that it is the energy dissipated into a resistance of 1 ohm.

We recall Parseval's theorem which states that if a function $f(t)$ is generally complex and if $F(j\omega)$ is the Fourier transform of $f(t)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega. \tag{12.2}$$

The energy in the resistance may therefore be written in the form

$$E = \int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega. \tag{12.3}$$

The function $|F(j\omega)|^2$ is called the *energy spectral density,* or simply the energy density, of $f(t)$. It is attributed the special symbol $\varepsilon_{ff}(\omega)$, that is,

$$\varepsilon_{ff}(\omega) \triangleq |F(j\omega)|^2 . \tag{12.4}$$

We note that its integral is equal to 2π times the signal energy

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)\, d\omega \tag{12.5}$$

hence the name 'spectral density'.

Given two signals $f_1(t)$ and $f_2(t)$, where $f_1(t)$ represent a current source and $f_2(t)$ the voltage that the current source produces across a resistance R of 1 ohm, we have

$$E = \int_{-\infty}^{\infty} f_1(t) f_2(t)\, dt. \tag{12.6}$$

Parseval's or Rayleigh's theorem is written

$$\int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(-j\omega) F_2(j\omega)\, d\omega. \tag{12.7}$$

If $f_1(t)$ and $f_2(t)$ are real

$$F_1(-j\omega) = F_1^*(j\omega), \quad F_2(-j\omega) = F^*(j\omega). \tag{12.8}$$

The *normalized cross-energy* or simply *cross-energy* is therefore given by

$$E_{f_1 f_2} = \int_{-\infty}^{\infty} f_1(t) f_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1^*(j\omega) F_2(j\omega)\, d\omega. \tag{12.9}$$

The function

$$\varepsilon_{f_1 f_2}(\omega) \triangleq F_1^*(j\omega) F_2(j\omega) \tag{12.10}$$

is called the *cross-energy spectral density.* The cross energy of the two signals is then given by

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varepsilon_{f_1 f_2}(\omega) d\omega. \tag{12.11}$$

Example 12.1 *Consider the ideal lowpass filter frequency response shown in Fig. 12.1. We have*

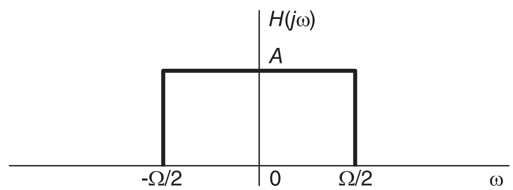

FIGURE 12.1
Ideal lowpass filter frequency response.

$$H(j\omega) = A\Pi_{\Omega/2}(\omega) = A\{u(\omega + \Omega/2) - u(\omega - \Omega/2)\}.$$

The filter's impulse response is given by

$$h(t) = \mathcal{F}^{-1}[H(j\omega)] = \frac{A\Omega}{2\pi} Sa(\Omega\, t/2).$$

The energy spectral density of $h(t)$ *is given by*

$$\varepsilon_{hh}(\omega) = |H(j\omega)|^2 = A^2 \Pi_{\Omega/2}(\omega).$$

We may evaluate the energy of $h(t)$ in a finite band of frequency, say, $\Omega/4 < |\omega| < \Omega/2$, as shown in Fig. 12.2.

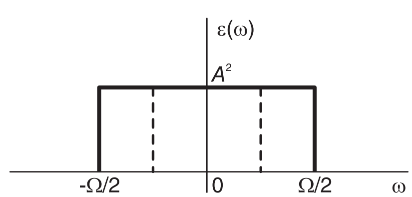

FIGURE 12.2
A frequency band of lowpass filter response.

$$E(\Omega/4,\ \Omega/2) = \frac{2}{2\pi}\int_{\Omega/4}^{\Omega/2} A^2 d\omega = \frac{A^2\Omega}{4\pi}. \tag{12.12}$$

The total energy of $h(t)$ is given by

$$E = \int_{-\infty}^{\infty} h^2(t)dt = \frac{A^2}{4\pi^2}\Omega^2 \int_{-\infty}^{\infty} Sa^2(\Omega t/2)dt. \tag{12.13}$$

It is easier, however, to evaluate the energy using Rayleigh's theorem. We write

$$E = \frac{2}{2\pi}\int_0^{\Omega/2} \varepsilon_{hh}(\omega)d\omega = \frac{A^2\Omega}{2\pi}. \tag{12.14}$$

We note that we have thus evaluated in passing the integral of the square of the sampling function. In particular, we found that

$$E = \frac{A^2\Omega^2}{4\pi^2}\int_{-\infty}^{\infty} Sa^2(\Omega t/2)dt = \frac{A^2\Omega}{2\pi}. \tag{12.15}$$

Substituting $\Omega t/2 = x$, we have

$$\int_{-\infty}^{\infty} Sa^2(x)dx = \pi. \tag{12.16}$$

Example 12.2 *Let*

$$v(t) = A\cos\omega_c t$$

and

$$v_T(t) = v(t)\,\Pi_{T/2}(t) = v(t)\{u(t+T/2) - u(t-T/2)\}.$$

Evaluate the energy spectral density of this truncated sinusoid shown in Fig. 12.3.

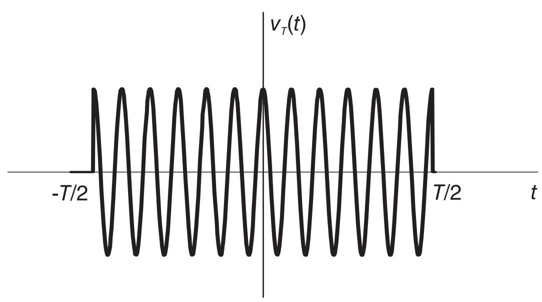

FIGURE 12.3
Truncated sinusoid.

We have

$$\Pi_{T/2}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} T\ Sa(\omega T/2)$$

$$V_T(j\omega) \triangleq \mathcal{F}[v_T(t)] = \frac{AT}{2}\{Sa[(\omega-\omega_c)T/2] + Sa[(\omega+\omega_c)T/2]\}$$

wherefrom the energy spectral density is given by

$$\varepsilon_{v_Tv_T}(\omega) = |V_T(j\omega)|^2 = (A^2T^2/4)\left\{Sa^2[(\omega-\omega_c)T/2] + Sa^2[(\omega+\omega_c)T/2] + 2Sa[(\omega-\omega_c)T/2]\,Sa[(\omega+\omega_c)T/2]\right\}$$

and is shown graphically in Fig. 12.4.

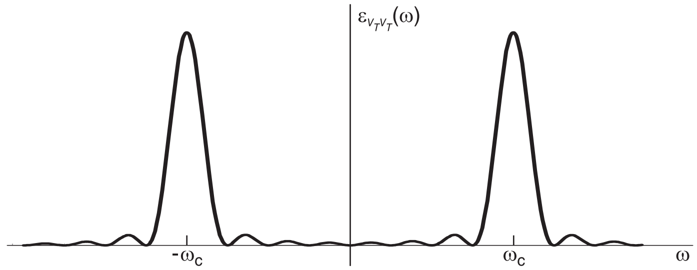

FIGURE 12.4
Energy spectral density.

12.2 Average, Energy and Power of Continuous-time Signals

The *average normalized power*, or simply *average power*, of a signal $f(t)$ is defined by

$$\overline{f^2}(t) \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2 dt. \tag{12.17}$$

The energy E, as seen above, is given by

$$E = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega. \tag{12.18}$$

A signal which has a finite energy E has an average power $\overline{f^2}(t)$ of zero. Such a signal is called an *energy signal*.

A *power signal* is one that has infinite energy and finite non-nil average power, i.e. $0 < \overline{f^2}(t) < \infty$. A periodic signal is a power signal. Its average power P is evaluated as its power over one period.

Let $f(t)$ be periodic of period T_0. Its average normalized power, or simply average power, is given by

$$P = \overline{f^2}(t) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |f(t)|^2 dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)f^*(t)dt. \tag{12.19}$$

From Parseval's relation for periodic functions

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2. \tag{12.20}$$

The average power of a periodic signal is thus given by the sum

$$P = \overline{f^2}(t) = \sum_{n=-\infty}^{\infty} |F_n|^2. \tag{12.21}$$

12.3 Discrete-Time signals

For discrete-time signals the energy and average power are similarly defined. If a sequence $f[n]$ has finite energy, defined as,

$$E = \sum_{n=-\infty}^{\infty} f^2[n] \tag{12.22}$$

it is called an *energy signal.*

If it has a finite average power, defined as

$$P = \lim_{M \to \infty} \frac{1}{2M} \sum_{n=-M}^{M} f^2[n] \tag{12.23}$$

it is called a power signal.

If the sequence is periodic with period M its average power over one period is

$$P = \frac{1}{M} \sum_{n=0}^{M-1} f^2 [n] \,. \tag{12.24}$$

An impulsive signal

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \delta(t - nT) \tag{12.25}$$

such as the one shown in Fig. 12.5 and which can be an ideal sampling of a continuous-time signal, is considered to be an energy signal if its average power defined as

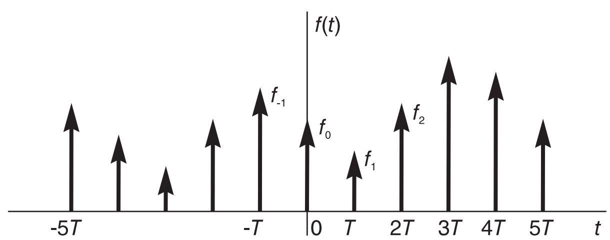

FIGURE 12.5
Impulsive signal.

$$\lim_{M \to \infty} \frac{1}{2MT} \sum_{n=-M}^{M} |f_n|^2 \tag{12.26}$$

is zero; otherwise it is a power signal.

12.4 Energy Signals

Let $f(t)$ and $g(t)$ be two real energy signals. We show that the Fourier transform of their cross-correlation function $r_{fg}(t)$ is equal to the cross spectral density $\varepsilon_{fg}(\omega)$.

We have already seen that correlation can be written as a convolution

$$r_{fg}(t) = \int_{-\infty}^{\infty} f(t + \tau) g(\tau) d\tau = f(t) * g(-t) \tag{12.27}$$

$$r_{fg}(-t) = r_{gf}(t) \,. \tag{12.28}$$

The Fourier transform of $r_{fg}(t)$ is therefore given by

$$R_{fg}(j\omega) = F(j\omega) G^*(j\omega) = \varepsilon_{fg}(\omega) \tag{12.29}$$

i.e. *the Fourier transform of the cross-correlation function of two energy signals is equal to their cross-energy spectral density.*

We note moreover, that if the functions $f(t)$ and $g(t)$ are complex then

$$r_{fg}(t) = \int_{-\infty}^{\infty} f(t+\tau)\, g^*(\tau)\, d\tau \tag{12.30}$$

$$R_{fg}(j\omega) \triangleq \mathcal{F}[r_{fg}(t)] = F(j\omega)\, G^*(j\omega) = \varepsilon_{fg}(\omega). \tag{12.31}$$

Moreover

$$r_{fg}(-t) = r_{fg}^*(t). \tag{12.32}$$

12.5 Auto-Correlation of Energy Signals

The Fourier transform of the autocorrelation function $r_{ff}(t)$ of an energy signal $f(t)$ is given by

$$R_{ff}(j\omega) = \mathcal{F}[r_{ff}(t)] = F(j\omega)\, F^*(j\omega) = |F(j\omega)|^2 = \epsilon_{ff}(\omega) \tag{12.33}$$

i.e.

$$r_{ff}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} |F(j\omega)|^2 = \varepsilon_{ff}(\omega) \tag{12.34}$$

$$\varepsilon_{ff}(\omega) = R_{ff}(j\omega) \tag{12.35}$$

so that the Fourier transform of the autocorrelation function of an energy signal is equal to the energy spectral density of the signal.

We note that the Fourier transform $F(j\omega)$ of a complex function $f(t)$ is not in general symmetric about origin, that is, $F(-j\omega) \neq F^*(j\omega)$. The energy spectral density $\varepsilon_{ff}(\omega) \triangleq |F(j\omega)|^2$ is real but not symmetric about the origin. Being real, however, its inverse transform is symmetric, that is, $r_{ff}(-t) = r_{ff}^*(t)$, as already established.

We note on the other hand that if the function $f(t)$ is real then $F(-j\omega) = F^*(j\omega)$ wherefrom the function $\varepsilon_{ff}(\omega) = |F(j\omega)|^2$ is even and its inverse transform $r_{ff}(t)$ is real (and even); $r_{ff}(-t) = r_{ff}(t)$.

Let $f(t)$ be generally complex. Writing

$$r_{ff,R}(t) \triangleq \Re[r_{ff}(t)], \quad r_{ff,I}(t) \triangleq \Im[r_{ff}(t)] \tag{12.36}$$

$$r_{ff,R}(t) = r_{ff,R}(-t) \tag{12.37}$$

$$r_{ff,I}(t) = -r_{ff,I}(-t) \tag{12.38}$$

$$\begin{aligned} \varepsilon_{ff}(\omega) &= \int_{-\infty}^{\infty} r_{ff}(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} [r_{ff,R}(t) + j r_{ff,I}(t)] (\cos\omega t - j\sin\omega t)\, dt \\ &= 2\int_{0}^{\infty} (r_{ff,R}(t)\cos\omega t + r_{ff,I}(t)\sin\omega t) dt \end{aligned} \tag{12.39}$$

$$\begin{aligned} r_{ff}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)\, e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)\cos\omega t\; d\omega + j\int_{\infty}^{\infty} \varepsilon_{ff}(\omega)\sin\omega t\; d\omega\right\} \end{aligned} \tag{12.40}$$

i.e.

$$r_{ff,R}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)\cos\omega t\; d\omega \tag{12.41}$$

$$r_{ff,I}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)\sin\omega t \, d\omega. \tag{12.42}$$

We note that

$$r_{ff}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon_{ff}(\omega)d\omega. \tag{12.43}$$

If the function $f(t)$ is real we have

$$r_{ff}(-t) = r_{ff}(t), \quad r_{ff,I}(t) = 0 \tag{12.44}$$

$$\varepsilon_{ff}(\omega) = |F(j\omega)|^2 = 2\int_0^{\infty} r_{ff}(t)\cos\omega t \, dt \tag{12.45}$$

$$r_{ff}(t) = \frac{1}{\pi}\int_0^{\infty} \varepsilon_{ff}(\omega)\cos\omega t \, d\omega \tag{12.46}$$

and

$$r_{ff}(t) \le r_{ff}(0) = E \tag{12.47}$$

E being the energy of $f(t)$.

Example 12.3 *Show that* $R_{ff}(j\omega) = \varepsilon_{ff}(\omega)$ *for the rectangular window*

$$f(t) = \Pi_T(t) = u(t+T) - u(t-T).$$

The transform of $f(t)$ *is*

$$F(j\omega) = 2T \; Sa(T\omega).$$

The spectral density is given by

$$\varepsilon_{ff}(\omega) = |F(j\omega)|^2 = 4T^2 Sa^2(T\omega).$$

The autocorrelation of $f(t)$ *is the triangle*

$$r_{ff}(t) = (2T - |t|)\,\Pi_{2t}(t) \triangleq 2T\Lambda_{2T}(t)$$

where, we recall, $\Lambda_x(t)$ *is a centered triangle of height unity and total base width* $2x$*. Its Fourier transform is*

$$R_{ff}(j\omega) \triangleq \mathcal{F}[r_{ff}(t)] = \varepsilon_{ff}(\omega).$$

The spectral density and autocorrelation function are shown in Fig. 12.6.

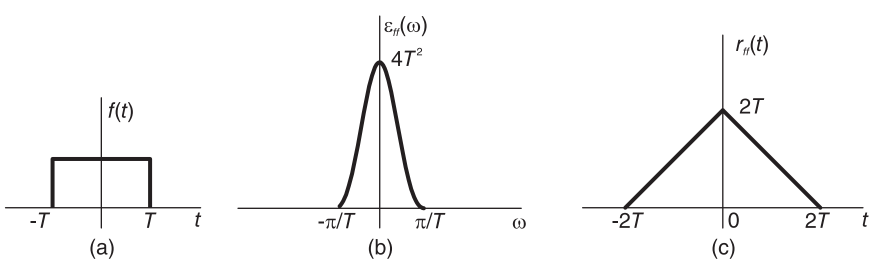

FIGURE 12.6
A rectangle, spectral density and autocorrelation function.

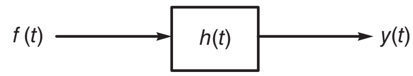

FIGURE 12.7
Linear system with input and output.

12.6 Energy Signal Through Linear System

Let an energy signal $f(t)$ be the input to a linear time invariant LTI system of impulse response $h(t)$, as shown in Fig. 12.7.

Let $r_{ff}(t)$ and $r_{yy}(t)$ be the autocorrelation of $f(t)$ and of $y(t)$, respectively. We have

$$R_{ff}(j\omega) = \mathcal{F}[r_{ff}(t)] = |F(j\omega)|^2 \quad (12.48)$$

$$R_{yy}(j\omega) = \mathcal{F}[r_{yy}(t)] = |Y(j\omega)|^2 . \quad (12.49)$$

Now

$$Y(j\omega) = F(j\omega) H(j\omega) \quad (12.50)$$

wherefrom

$$R_{yy}(j\omega) = |F(j\omega)|^2 |H(j\omega)|^2 \quad (12.51)$$

i.e.

$$R_{yy}(j\omega) = R_{ff}(j\omega) |H(j\omega)|^2 = R_{ff}(j\omega) H(j\omega) H^*(j\omega) . \quad (12.52)$$

Hence

$$\varepsilon_{yy}(\omega) = \varepsilon_{ff}(\omega) |H(j\omega)|^2 . \quad (12.53)$$

Moreover

$$\mathcal{F}^{-1}[H^*(j\omega)] = h(-t) \quad (12.54)$$

we have

$$r_{yy}(t) = r_{ff}(t) * h(t) * h(-t) \quad (12.55)$$

i.e. the autocorrelation of the system response is the convolution of the input signal autocorrelation with the convolution $h(t) * h(-t)$.

12.7 Impulsive and Discrete-Time Energy signals

Let $f_s(t)$ be a signal formed of equidistant impulses such as the signal

$$f_s(t) = \ldots + f[-1]\,\delta(t+T) + f[0]\,\delta(t) + f[1]\,\delta(t-T) + \ldots \quad (12.56)$$

$$= \sum_{n=-\infty}^{\infty} f[n]\,\delta(t-nT) \quad (12.57)$$

shown in Fig. 12.8 (a).

We may view the impulsive signal $f_s(t)$ as the result of sampling a continuous-time signal $f_c(t)$ with a sampling interval of T seconds.

$$f_s(t) = f_c(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} f_c(nT)\,\delta(t-nT) . \quad (12.58)$$

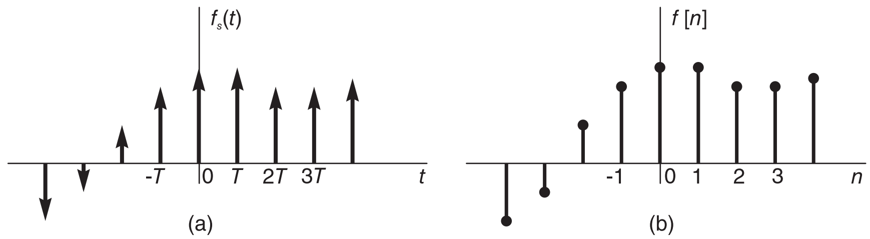

FIGURE 12.8
Signal with equidistant impulses and discrete-time signal counterpart.

Associated with $f_c(t)$ and $f_s(t)$ we also have a discrete-time function, namely, the sequence $f[n] = f_c(nT)$ shown in Fig. 12.8 (b). The energy of the signal $f_s(t)$ as well as that of $f[n]$ are defined by the summation

$$E = \sum_{n=-\infty}^{\infty} |f[n]|^2 . \tag{12.59}$$

If the energy is finite then the signal $f_s(t)$ and the sequence $f[n]$ are energy signals. The auto-correlation of the signal $f_s(t)$ can be obtained by evaluating the auto-correlation $r_{ff}[n]$ of the corresponding sequence $f[n]$. In fact the auto-correlation of $f_s(t)$ is given by

$$\begin{aligned}
r_{f_s f_s}(t) &= \int_{-\infty}^{\infty} f_s(\tau) f_s(t+\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[m]\, \delta(\tau - mT) \sum_{i=-\infty}^{\infty} f[i] \delta(t+\tau - iT)\, d\tau \\
&= \int_{-\infty}^{\infty} \sum_{m} \sum_{i} f[m] f[i]\, \delta(\tau - mT)\, \delta(t+\tau - iT)\, d\tau \\
&= \sum_{m} \sum_{i} f[m] f[i] \int_{\infty}^{\infty} \delta(\tau - mT) \delta(t+\tau - iT)\, d\tau \\
&= \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f[m] f[i]\, \delta(t - (i-m)T) .
\end{aligned}$$

Letting $i - m = n$ we have

$$r_{f_s f_s}(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m] f[m+n]\, \delta(t - nT) . \tag{12.60}$$

Interchanging the order of summations

$$r_{f_s f_s}(t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[m]\, f[m+n]\, \delta(t - nT) = \sum_{n=-\infty}^{\infty} \rho_n \delta(t - nT) \tag{12.61}$$

where

$$\rho_n = \sum_{m=-\infty}^{\infty} f[m] f[m+n] . \tag{12.62}$$

On the other hand the discrete auto-correlation of the corresponding sequence $f[n]$ is given by

$$r_{ff}[n] = \sum_{m=-\infty}^{\infty} f[m] f[n+m] . \tag{12.63}$$

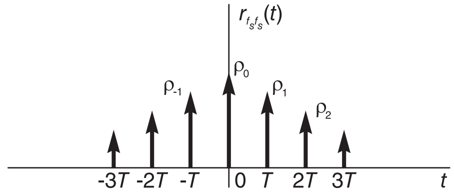

FIGURE 12.9
Auto-correlation of an impulsive signal.

Hence

$$\rho_n = r_{ff}\left[n\right]. \tag{12.64}$$

The autocorrelation $r_{f_sf_s}(t)$ is represented graphically in Fig. 12.9.

The auto-correlation of the impulsive signal $f_s(t)$ is therefore a one-to-one correspondence to the discrete auto-correlation of the corresponding discrete-time-sequence $f[n]$. It can be evaluated by simply effecting a discrete auto-correlation of the discrete sequence $f[n]$, followed by converting the resulting sequence $r_{ff}[n]$ into the corresponding impulsive function, which is the auto-correlation function $r_{f_sf_s}(t)$ of the function $f_s(t)$. The same approach can be used for evaluating the cross-correlation of two impulsive functions $f_s(t)$ and $g_s(t)$.

The Fourier transform of $f_s(t)$ is given by

$$F_s(j\omega) = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} f[n]\,\delta(t-nT)\right] = \frac{1}{T}\sum_{n=-\infty}^{\infty} F_c\left(j\omega + j\frac{2\pi n}{T}\right). \tag{12.65}$$

This is equal to the Fourier transform $F\left(e^{j\Omega}\right)$ of the discrete-time counterpart, the sequence $f[n]$ with $\Omega = \omega T$.

$$F\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} = F_s(j\omega)\big|_{\omega=\Omega/T} = F_s\left(j\frac{\Omega}{T}\right). \tag{12.66}$$

The energy density $\varepsilon_{f_sf_s}(\omega)$ of the signal $f_s(t)$ is given by

$$\varepsilon_{f_sf_s}(\omega) = \left|F_s(j\omega)\right|^2 \tag{12.67}$$

and is therefore periodic of a period $2\pi/T$. Similarly the energy density of the sequence $f[n]$ is given by

$$\varepsilon_{ff}(\Omega) = \left|F\left(e^{j\Omega}\right)\right|^2 \tag{12.68}$$

and is periodic with a period 2π. The autocorrelation $r_{f_sf_s}(t)$ maybe written as the convolution:

$$r_{f_sf_s}(t) = f_s(t) \star f_s(t) = f_s(t) * f_s(-t) \tag{12.69}$$

$$R_{f_sf_s}(j\omega) = F_s(j\omega)\,F_s^*(j\omega) = \left|F_s(j\omega)\right|^2 = \varepsilon_{f_sf_s}(\omega) \tag{12.70}$$

$$r_{ff}[n] = f[n] \star f[n] = f[n] * f[-n] \tag{12.71}$$

$$R_{ff}\left(e^{i\Omega}\right) = F\left(e^{j\Omega}\right)F\left(e^{-j\Omega}\right) = \left|F\left(e^{j\Omega}\right)\right|^2 = \varepsilon_{ff}(\Omega). \tag{12.72}$$

The transform of the energy spectral density, is therefore given by

$$\varepsilon_{f_sf_s}(\omega) = R_{f_sf_s}(j\omega) = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \rho_n\delta(t-nt)\right] = \sum_{n=-\infty}^{\infty} \rho_n e^{-j\omega nT} \tag{12.73}$$

and

$$\varepsilon_{ff}(\Omega) = R_{ff}\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} r_{ff}[n]e^{-j\Omega n}. \tag{12.74}$$

Since $f(t)$ is real we have $r_{ff}[-n] = r_{ff}[n]$ and $r_{f_sf_s}(-t) = r_{f_sf_s}(t)$, i.e., $\rho_{-n} = \rho_n$.

$$\varepsilon_{f_sf_s}(\omega) = \rho_0 + 2\sum_{n=1}^{\infty} \rho_n \cos nT\omega = r_{ff}[0] + 2\sum_{n-1}^{\infty} r_{ff}[n] \cos nT\omega \tag{12.75}$$

and

$$\varepsilon_{ff}(\Omega) = r_{ff}[0] + 2\sum_{n=1}^{\infty} r_{ff}[n] \cos n\,\Omega. \tag{12.76}$$

Example 12.4 *Let*

$$f_s(t) = \delta(t - T) + 2\delta(t - 2T).$$

The signal is shown in Fig. 12.10 (a).

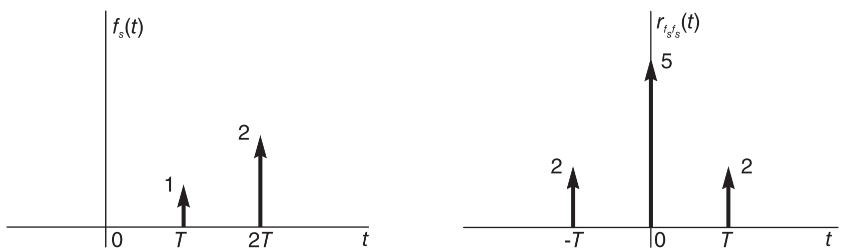

FIGURE 12.10
Impulsive signal and its autocorrelation.

Its autocorrelation is shown in Fig. 12.10 (b). The autocorrelation may be found by evaluating the auto correlation of the corresponding sequence $f[n] = \delta[n-1] + 2\delta[n-2]$. *We have*

$$\rho_n = r_{ff}[n] = \sum_{n=-\infty}^{\infty} f[n+m]f[m] = 2\delta[n+1] + 5\delta[n] + 2\delta[n-1].$$

The sequence $f[n]$ *and its auto correlation* $r_{ff}[n] = \rho_n$ *are shown in Fig. 12.11.*

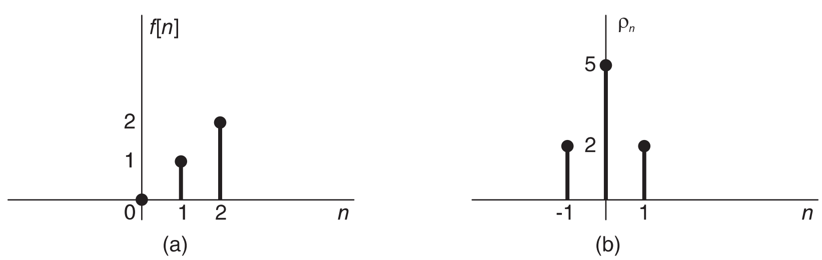

FIGURE 12.11
A sequence and its autocorrelation.

$$r_{f_sf_s}(t) = 5\delta(t) + 2\delta(t+T) + 2\delta(t-T)$$
$$\varepsilon_{f_sf_s}(\omega) = R_{f_sf_s}(j\omega) = 5 + 2e^{j\omega T} + 2e^{-j\omega T} = 5 + 4\cos T\omega.$$

Alternatively, we have

$$F_s(j\omega) = e^{-j\omega t} + 2e^{-j\omega t} = (\cos\omega T + 2\cos 2\omega T) - j(\sin\omega T + 2\sin 2\omega T)$$

$$\varepsilon_{f_s f_s}(\omega) = |F_s(j\omega)|^2 .$$

Similarly $\varepsilon_{ff}(\Omega) = R_{ff}\left(e^{j\Omega}\right) = 5 + 4\cos\Omega.$

Example 12.5 *Let*

$$f_c(t) = \begin{cases} t/10, & 0 \le t \le 30 \\ 6 - t/10, & 30 \le t \le 60. \end{cases}$$

Evaluate the sampled function $f_s(t)$, the discrete-time function $f[n]$ and their auto-correlations, assuming a sampling interval of $T = 10$ sec. We have

$$f_s(t) = \delta(t-T) + 2\delta(t-2T) + 3\delta(t-3T) + 2\delta(t-4T) + \delta(t-5T)$$

$$f[n] = f_c(nT) = f_c(10n) = \begin{cases} n, & 0 \le n \le 3 \\ 6-n, & 3 \le n \le 6 \end{cases}$$

$$\begin{aligned}\rho_n = r_{ff}[n] &= \delta[n+4] + 4\,\delta[n+3] + 10\,\delta[n+2] + 16\,\delta[n+1] \\ &+ 19\,\delta[n] + 16\,\delta[n-1] + 10\,\delta[n-2] + 4\,\delta[n-3] + \delta[n-4]\,.\end{aligned}$$

The sequence $f[n]$ and its autocorrelation $\rho[n] = r_{ff}[n]$ are shown in Fig. 12.12 (a) and (b), respectively.

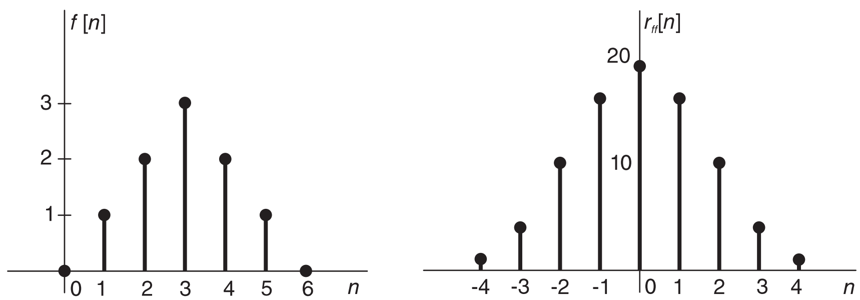

FIGURE 12.12
Sequence $f[n]$ and its autocorrelation.

The corresponding impulsive auto-correlation function $r_{f_s f_s}(t)$ is deduced thereof to be

$$\begin{aligned}r_{f_s f_s}(t) &= \delta(t+4T) + 4\delta(t+3T) + 10\delta(t+2T) + 16\delta(t+T) \\ &+ 19\delta(t) + 16\delta(t-T) + 10\delta(t-2T) + 4\delta(t-3T) + \delta(t-4T)\end{aligned}$$

$$\begin{aligned}\varepsilon_{f_s f_s}(\omega) &= R_{f_s f_s}(j\omega) \\ &= 19 + 32\cos T\omega + 20\cos 2T\omega + 8\cos 3T\omega + 2\cos 4T\omega \\ &= 19 + 32\cos 10\omega + 20\cos 20\omega + 8\cos 30\omega + 2\cos 40\omega\end{aligned}$$

$$\varepsilon_{ff}(\Omega) = R_{ff}\left(e^{i\Omega}\right) = 19 + 32\cos\Omega + 20\cos 2\Omega + 8\cos 3\Omega + 2\cos 4\Omega.$$

The energy spectral density $\varepsilon_{ff}(\Omega)$ of the sequence $f[n]$ is shown in Fig. 12.13. Alternatively,

$$F_s(j\omega) = e^{-j\omega T} + 2e^{-j2\omega T} + 3e^{-j3\omega T} + 2e^{-j4\omega T} + e^{-j5\omega T}$$

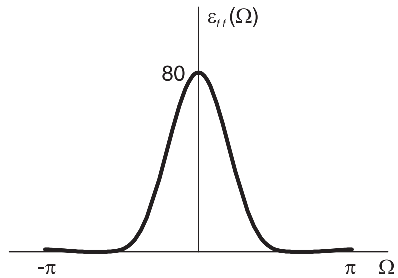

FIGURE 12.13
Energy spectral density.

$$\varepsilon_{f_s f_s}(\omega) = |F_s(j\omega)|^2 = F_s(j\omega)\, F_s^*(j\omega)\,.$$

Letting

$$z = e^{j\omega T}, \quad z^* = e^{-j\omega T} = z^{-1}.$$

We have, with $z = e^{j\Omega}$,

$$\begin{aligned} \varepsilon_{f_s f_s}(\omega) &= \left(z^{-1} + 2z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5}\right) \\ &\quad \left(z + 2z^2 + 3z^3 + 2z^4 + z^5\right) \\ &= 19 + 16z^{-1} + 10z^{-2} + 4z^{-3} + z^{-4} + 16z + 10z^2 + 4z^3 + z^4 \\ &= 19 + 32\cos\omega T + 20\cos 2\omega T + 8\cos 3\omega T + 2\cos 4\omega T = R_{f_s f_s}(j\omega) \\ \varepsilon_{ff}(\Omega) &= 19 + 32\cos\Omega + 20\cos 2\Omega + 8\cos 3\Omega + 2\cos 4\Omega = R_{ff}\left(e^{j\Omega}\right). \end{aligned}$$

12.8 Powers Signals

We have seen that a power signal has a finite average power

$$0 < \overline{f^2}(t) < \infty, \tag{12.77}$$

where

$$\overline{f^2}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt \tag{12.78}$$

and that a periodic signal is a power signal having an average power evaluated over one period

$$P = \overline{f^2}(t) = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2\,. \tag{12.79}$$

In the following the cross and auto-correlation of such signals are defined.

12.9 Cross-Correlation

Let $f(t)$ and $g(t)$ be two real power signals. The cross-correlation $r_{fg}(t)$ is given by

$$r_{fg}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+\tau) g(\tau)\, d\tau \tag{12.80}$$

$$r_{fg}(-t) = r_{gf}(t) \tag{12.81}$$

as is the case for energy signals. If $f(t)$ and $g(t)$ are complex then

$$r_{fg}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+\tau)\, g^*(\tau)\, d\tau \tag{12.82}$$

$$r_{fg}(-t) = r_{gf}^*(t) \tag{12.83}$$

$$r_{ff}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\tau) f(t+\tau)\, d\tau \tag{12.84}$$

$$r_{ff}(-t) = r_{ff}(t) \tag{12.85}$$

and

$$r_{ff}(0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt = \overline{f^2}(t). \tag{12.86}$$

12.9.1 Power Spectral Density

For a real power signal $f(t)$ the power spectral density denoted by $S_{ff}(\omega)$ is by definition the Fourier transform of the autocorrelation function.

$$S_{ff}(\omega) = \mathcal{F}[r_{ff}(t)] = R_{ff}(j\omega). \tag{12.87}$$

Since $r_{ff}(t)$ is real and even its transform $S_{ff}(\omega)$ is real and even. We have

$$S_{ff}(\omega) = 2\int_0^\infty r_{ff}(t)\cos\omega t\; dt \tag{12.88}$$

and

$$r_{ff}(t) = \frac{1}{\pi}\int_0^\infty S_{ff}(\omega)\cos\omega t\; d\omega. \tag{12.89}$$

Let

$$f_T(t) = f(t)\,\Pi_T(t) = f(t)\{u(t+T) - u(t-T)\} \tag{12.90}$$

that is, $f_T(t)$ is a truncation of $f(t)$.

We have

$$F_T(j\omega) = \mathcal{F}[f_T(t)] = \int_{-T}^{T} f(t) e^{-j\omega t} dt. \tag{12.91}$$

The average power density over the interval $(-T,\ T)$ is the energy over the interval divided by the duration $2T$. Denoting it by $S_T(\omega)$ we have

$$S_T(\omega) \triangleq \frac{1}{2T}|F_T(j\omega)|^2. \tag{12.92}$$

It can be shown that $S_{ff}(\omega)$ is the limit as T tends to infinity of $S_T(\omega)$

$$S_{ff}(\omega) = \lim_{T\to\infty} S_T(\omega) = \lim_{T\to\infty}\frac{1}{2T}|F_T(j\omega)|^2. \tag{12.93}$$

In fact

$$\begin{aligned} S_{ff}(\omega) &= \mathcal{F}[r_{ff}(t)] = \mathcal{F}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_T(t+\tau) f_T(\tau)\, d\tau\right] \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty}\int_{-T}^{T} f_T(t+\tau) f_T(\tau) d\tau\; e^{-j\omega t} dt \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_T(\tau)\int_{-\infty}^{\infty} f_T(t+\tau)\, e^{-j\omega t} dt\; d\tau. \end{aligned} \tag{12.94}$$

Let

$$t + \tau = x \tag{12.95}$$

$$
\begin{aligned}
S_{ff}(\omega) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_T(\tau) \int_{-\infty}^{\infty} f_T(x)\, e^{-j\omega(x-\tau)} dx\, d\tau \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_T(\tau) e^{j\omega\tau} F_T(j\omega)\, d\tau \\
&= \lim_{T\to\infty} \frac{1}{2T} F_T(j\omega) \int_{-T}^{T} f(\tau) e^{j\omega\tau} d\tau = \lim_{T\to\infty} \frac{1}{2T} F_T(j\omega)\, F_T^*(j\omega) \\
&= \lim_{T\to\infty} \frac{1}{2T} |F_T(j\omega)|^2 = \lim_{T\to\infty} S_T(\omega)\,.
\end{aligned} \tag{12.96}
$$

12.10 Power Spectrum Conversion of a Linear System

Let $f(t)$ be a power signal applied to the input of a linear time invariant LTI system the impulse response of which $h(t)$ is an energy signal. The system response may be written

$$y(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau)\, d\tau. \tag{12.97}$$

Let $r_{ff}(t)$ and $S_{ff}(\omega)$ be the autocorrelation and spectral density respectively of the input $f(t)$. The autocorrelation of the output signal $y(t)$ is given by

$$
\begin{aligned}
r_{yy}(t) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(\tau) y(t+\tau)\, d\tau \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h(u) f(\tau-u)\, du \int_{-\infty}^{\infty} h(x) f(t+\tau-x)\, dx\, dt.
\end{aligned} \tag{12.98}
$$

Interchanging the order of integration

$$
\begin{aligned}
r_{yy}(t) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(x) \int_{-T}^{T} f(\tau-u) f(t+\tau-x)\, d\tau\, dx\, du \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(x) \int_{-T-u}^{T-u} f(\alpha) f(\alpha+u+t-x)\, d\tau\, dx\, du \\
&= \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(x) r_{ff}(u+t-x)\, dx\, du.
\end{aligned} \tag{12.99}
$$

We note that the second integral is a convolution. Writing

$$z(u+t) = \int_{-\infty}^{\infty} h(x) r_{ff}(u+t-x)\, dx = h(t) * r_{ff}(u+t) \tag{12.100}$$

i.e.

$$z(t) = h(t) * r_{ff}(t) \tag{12.101}$$

we have

$$r_{yy}(t) = \int_{-\infty}^{\infty} h(u) z(u+t)\, du = r_{zh}(t) = z(t) * h(-t) = r_{ff}(t) * h(t) * h(-t)\,. \tag{12.102}$$

We conclude that the system response $y(t)$ is a power signal the autocorrelation $r_{yy}(t)$ of which is the convolution of the input signal autocorrelation $r_{ff}(t)$ with the function $h(t) * h(-t)$ that is, the convolution of $h(t)$ with is reflection. Moreover,

$$S_{yy}(\omega) = \mathcal{F}[r_{yy}(t)] = \mathcal{F}[r_{ff}(t)] \cdot H(j\omega)\, H^*(j\omega) = S_{ff}(\omega) |H(j\omega)|^2\,. \tag{12.103}$$

We conclude that the time domain convolution $y(t) = f(t) * h(t)$ leads to the power spectral density transformation

$$S_{yy}(\omega) = S_{ff}(\omega)\,|H(j\omega)|^2 \tag{12.104}$$

and that more generally, the convolution $y(t) = f(t) * x(t)$ of a power signal $f(t)$ and an energy signal $x(t)$ leads to the power spectral density transformation

$$S_{yy}(\omega) = S_{ff}(\omega)\,|X(j\omega)|^2. \tag{12.105}$$

In the case of input white noise for example

$$S_{ff}(\omega) = 1 \tag{12.106}$$

wherefrom $r_{ff}(t) = \delta(t)$ and $S_{yy}(\omega) = |H(j\omega)|^2$, i.e. the power density of the system response is equal to the energy density of the impulse response $h(t)$.

Example 12.6 *Let $f(t) = K$, where K is a constant. The autocorrelation of $f(t)$ given by*

$$r_{ff}(t) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} K^2 dt = K^2$$

is a constant, and

$$S_{ff}(\omega) = \mathcal{F}[r_{ff}(t)] = R_z zj\omega = 2\pi K^2 \delta(\omega)$$

as shown in Fig. 12.14.

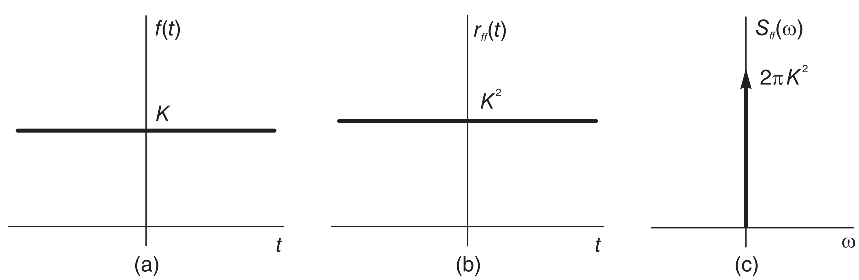

FIGURE 12.14
A constant, autocorrelation and power spectral density.

The power by direct evaluation is $P = K^2$ and, alternatively,

$$P = \overline{f^2}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ff}(\omega)d\omega = K^2.$$

Note that functions that are absolutely integrable such $e^{-t}u(t)$ have finite energy and thus represent energy signals whereas functions such as the step function and unity represent power signals.

Example 12.7 *Evaluate the autocorrelation and spectral density of the signal*

$$f(t) = Ku(t).$$

The signal is shown in Fig. 12.15(a).

$$r_{ff}(t) = \lim_{T\to\infty} \frac{K^2}{2T}\int_{-T}^{T} u(\tau)\,u(t+\tau)\,d\tau.$$

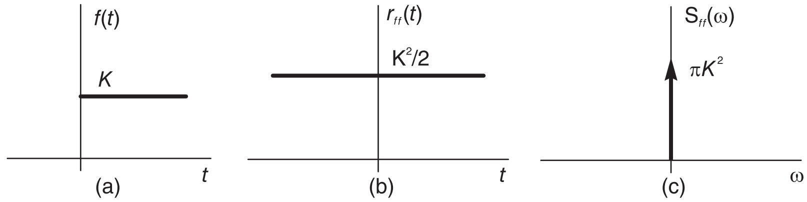

FIGURE 12.15
Unit step function, autocorrelation and power spectral density.

Consider the integral

$$I = \int_{-T}^{T} u(\tau)u\,(t+\tau)\,d\tau$$

and the case $t > 0$. *We have*

$$I = \int_{o}^{T} d\tau = T$$

and

$$r_{ff}\,(t) = \lim_{T\to\infty} \frac{K^2}{2T} I = \frac{K^2}{2}, \quad t > 0.$$

For $t < 0$ *we can use the symmetry property*

$$r_{ff}\,(-t) = r_{ff}\,(t) = K^2/2$$

wherefrom

$$r_{ff}\,(t) = K^2/2, \quad \forall t$$

and

$$S_{ff}\,(\omega) = R_{ff}(j\omega) = \pi K^2 \delta\,(\omega)\,.$$

The autocorrelation and spectral density are shown in Fig. 12.15(b) and (c), respectively.

12.11 Impulsive and Discrete-Time Power Signals

Let $f\,(t)$ be the impulsive function

$$f_s\,(t) = \sum_{n=-\infty}^{\infty} f\,[n]\delta\,(t - nT)\,. \tag{12.107}$$

If the average power of $f\,(t)$ is finite and not zero, that is,

$$0 < \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} |f\,[n]|^2 < \infty \tag{12.108}$$

then $f\,(t)$ is a power signal. As noted earlier $f_s\,(t)$ may be the result of ideal sampling of a continuous-time function $f_c\,(t)$

$$f_s\,(t) = \sum_{n=-\infty}^{\infty} f_c\,(nT)\delta\,(t - nT)\,. \tag{12.109}$$

The discrete-time representation of the same signal is the sequence $f[n]$ defined by $f[n] = f_c(nT)$. The autocorrelation of $f_s(t)$ is given by

$$r_{f_s f_s}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_s(\tau) f_s(t+\tau)\, d\tau. \tag{12.110}$$

As in the case of impulsive and discrete-time energy signals it can be shown that

$$r_{f_s f_s}(t) = \sum_{n=-\infty}^{\infty} \rho_n \delta(t - nT) \tag{12.111}$$

where

$$\rho_n = \lim_{M\to\infty} \frac{1}{2MT} \sum_{m=-M}^{M-1} f[m] f[m+n]. \tag{12.112}$$

The power density is given by

$$\begin{aligned} S_{f_s f_s}(\omega) &= \mathcal{F}\left[r_{f_s f_s}(t)\right] \triangleq R_{f_s f_s}(j\omega) = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \rho_n \delta(t - nT)\right] \\ &= \sum_{n=-\infty}^{\infty} \rho_n e^{-jnT\omega} = \rho_0 + 2\sum_{n=1}^{\infty} \rho_n \cos nT\omega. \end{aligned} \tag{12.113}$$

For the sequence $f[n]$ the autocorrelation is given by

$$r_{ff}[n] = \lim_{M\to\infty} \frac{1}{2M} \sum_{m=-M}^{M-1} f[m] f[n+m] \tag{12.114}$$

so that

$$\rho_n = \frac{1}{T} r_{ff}[n] \tag{12.115}$$

$$\begin{aligned} S_{ff}(\Omega) &= \mathcal{F}\left[r_{ff}[n]\right] = R_{ff}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} r_{ff}[n] e^{-j\Omega n} \\ &= r_{ff}[0] + 2\sum_{n=1}^{\infty} r_{ff}[n] \cos \Omega\, n. \end{aligned} \tag{12.116}$$

12.12 Periodic Signals

Let a real signal $f(t)$ be periodic of period T. Its autocorrelation $r_{ff}(t)$ is periodic defined by

$$\begin{aligned} r_{ff}(t) &= \frac{1}{T} \int_0^T f(\tau) f(t+\tau)\, d\tau = \frac{1}{T} \int_0^T f(\tau) \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 (t+\tau)} d\tau \\ &= \frac{1}{T} \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 \tau} \int_0^T f(\tau) e^{jn\omega_0 \tau} d\tau = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} F_n^* \end{aligned} \tag{12.117}$$

i.e.

$$r_{ff}(t) = \sum_{n=-\infty}^{\infty} |F_n|^2 e^{jn\omega_0 t}, \quad \omega_0 = 2\pi/T \tag{12.118}$$

which has the form of a Fourier series expansion having as coefficients $|F_n|^2$. We can therefore write

$$|F_n|^2 = \frac{1}{T} \int_T r_{ff}(t) e^{-jn\omega_0 t} dt \tag{12.119}$$

$$r_{ff}(t) = \sum_{n=-\infty}^{\infty} |F_n|^2 \cos n\omega_0\, t \tag{12.120}$$

$$r_{ff}(t) = |F_0|^2 + 2\sum_{n=1}^{\infty} |F_n|^2 \cos n\, \omega_0 t. \tag{12.121}$$

The power spectral density is given by

$$S_{ff}(\omega) = R_{ff}(j\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0). \tag{12.122}$$

The average power of $f(t)$ is given by

$$P = \overline{f^2(t)} = r_{ff}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(j\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ff}(\omega) d\omega. \tag{12.123}$$

Moreover,

$$P = \frac{1}{T} \int_T f^2(t)\, dt = \sum_{n=-\infty}^{\infty} |F_n|^2. \tag{12.124}$$

Example 12.8 *Evaluate the power, the spectral density and autocorrelation function of the signal* $f(t) = A\cos\omega_0 t$ *where* $\omega_0 = 2\pi/T$. *We have*

$$P = \frac{1}{T} \int_0^T A^2 \cos^2\omega_0 t\ dt = \frac{A^2}{T} \times \frac{1}{2} \int_0^T (\cos 2\omega_0 t + 1) dt = A^2/2.$$

The evaluation of the average power of a sinusoid is often needed. It is worth while remembering that the average power of a sinusoid of amplitude A *is simply* $A^2/2$.

We also note that the Fourier series coefficients of the expansion

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

are given by

$$F_n = \begin{cases} A/2, & n = \pm 1 \\ 0, & \textit{otherwise} \end{cases}$$

wherefrom

$$P = \overline{f^2(t)} = \sum |F_n|^2 = 2 \times A^2/4 = A^2/2$$

$$S_{ff}(\omega) = 2\pi \sum |F_n|^2 \delta(\omega - n\omega_0) = \pi \frac{A^2}{2} \{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\}$$

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi A^2}{2} \{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\} d\omega = A^2/2$$

$$r_{ff}(t) = |F_0|^2 + 2\sum_{1}^{\infty} |F_n|^2 \cos n\, \omega_0 t = (A^2/2)\cos\omega_0 t.$$

We note, moreover, that

$$R_{ff}(j\omega) = \frac{\pi A^2}{2} \{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\} = S_{ff}(\omega).$$

12.12.1 Response of an LTI System to a Sinusoidal Input

let $x(t) = \sin(\beta t + \theta)$ be the input to an LTI system. We evaluate the power spectral density at the input and output of the system.
The power spectral density of the input is

$$S_{xx}(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(\omega - n\omega_0). \tag{12.125}$$

where $\omega_0 = \beta$. The power spectral density of the output is

$$S_{yy}(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |Y_n|^2 \delta(\omega - n\omega_0) == 2\pi \sum_{n=-\infty}^{\infty} |X_n|^2 |H(jn\beta)|^2 \delta(\omega - n\beta). \tag{12.126}$$

The average power of the input $x(t)$ is

$$P = \overline{x^2(t)} = \sum_{n=-\infty}^{\infty} |X_n|^2 = A^2/2. \tag{12.127}$$

and that of the output is

$$P = \overline{y^2(t)} = \sum_{n=-\infty}^{\infty} |Y_n|^2 = (A^2/2)|H(jn\beta)|^2 \tag{12.128}$$

Example 12.9 *The signal $x(t) = A\sin(\beta t)$, with $A = 1$ and $\beta = \pi$, is applied to the input of an LTI system of impulse response $h(t) = \Pi_{0.5}(t)$. Is the system response $y(t)$ an energy or power signal? Evaluate the energy and power, and the spectral density at the system input and output.*
The input signal $x(t)$ and response $y(t)$ have infinite energy and are hence power signals. since their energy is infinite. The spectral densities are

$$S_x(\omega) = (\pi/2)[\delta(\omega - \pi) + \delta(\omega + \pi)]$$

and

$$S_y(\omega) = S_x(\omega)|H(j\omega)|^2 = \frac{\pi}{2} Sa^2(\pi/2)[\delta(\omega - \pi) + \delta(\omega + \pi)] = 0.637[\delta(\omega - \pi) + \delta(\omega + \pi)]$$

The input power is

$$P_x = \overline{x^2(t)} = \frac{1}{2\pi}\int_{\infty}^{\infty} S_x(\omega) d\omega = 0.5$$

The output power is

$$P_y = \overline{y^2(t)} = \frac{1}{2\pi}\int_{\infty}^{\infty} S_y(\omega) d\omega = 0.203$$

Alternatively, note that the input sinusoid Amplitude is $A = 1$ and its power is $P_x = \overline{x^2(t)} = A^2/2 = 0.5$. The output is $y(t) = A|H(j\pi)|\sin(\beta t + arg[H(j\pi)]) = B\sin(\pi t + \theta)$, where $B = 0.6366$ and $\theta = -\pi/2$, and its power is $P_y = \overline{y^2(t)} = B^2/2 = 0.203$.

12.13 Power Spectral Density of an Impulse Train

Consider the impulse train shown in Fig. 12.16(a).

$$x(t) = \rho_T(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT). \tag{12.129}$$

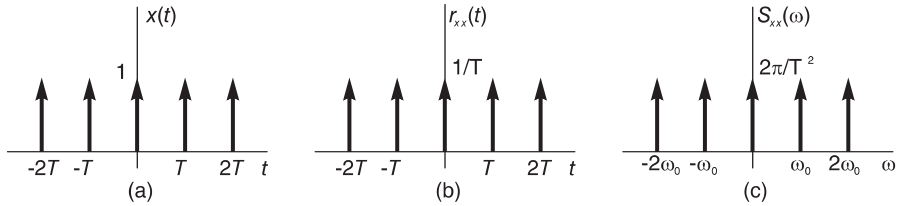

FIGURE 12.16
Impulse train, autocorrelation and power spectral density.

To evaluate the power spectral density of the impulse train we may proceed by applying the correlation definition directly over one period.

$$r_{xx}(t) = \frac{1}{T}\int_{-T/2}^{T/2} \delta(\tau)\delta(t+\tau)\,d\tau = \frac{1}{T}\delta(t), \quad -T/2 \le t \le T/2 \tag{12.130}$$

that is, $r_{xx}(t)$ is an impulse train of period T and impulses of intensity $1/T$

$$r_{xx}(t) = \frac{1}{T}\sum \delta(t - nT) = \frac{1}{T}\rho_T(t). \tag{12.131}$$

The power spectral density with $\omega_0 = 2\pi/T$ is given by

$$S_{xx}(\omega) = R_{xx}(j\omega) = \frac{1}{T}\omega_0\rho_{\omega_0}(\omega) = \frac{2\pi}{T^2}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0). \tag{12.132}$$

Alternatively, $X_n = 1/T$ and

$$S_{xx}(\omega) = 2\pi\sum_{n=-\infty}^{\infty}|X_n|^2\delta(\omega - n\omega_0) = \frac{2\pi}{T^2}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0). \tag{12.133}$$

Example 12.10 *Let $v(t)$ be the periodic ramp shown in Fig. 12.18. Evaluate the power spectral density. We have found in Chapter 2 that the fourier series coefficients are*

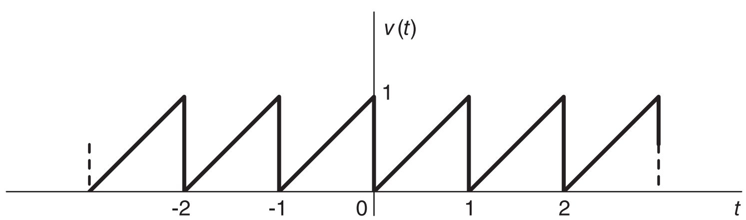

FIGURE 12.17
Periodic ramp.

$$V_n = \begin{cases} A/2, & n = 0 \\ jA/(2\pi n), & n \ne 0 \end{cases}$$

where $A = 1$ *and* $\omega_0 = 2\pi$. *Hence*

$$S_{vv}(\omega) = 2\pi\sum_{n=-\infty}^{\infty}|V_n|^2\,\delta(\omega - n\omega_0) = \left(\pi A^2/2\right)\delta(\omega) + \sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty}\frac{A^2}{2\pi n^2}\delta(\omega - n\omega_0)$$

$$r_{vv}(t) = V_0^2 + 2\sum_{n=1}^{\infty} |V_n|^2 \cos n\omega_0 t = 1/4 + \sum_{n=1}^{\infty} \left(\frac{1}{2\pi^2 n^2}\right) \cos n\omega_0 t.$$

A direct evaluation of the periodic autocorrelation of the periodic ramp $v(t)$ by the usual shift-multiply-integrate process as shown in Fig. **??** *we obtain*

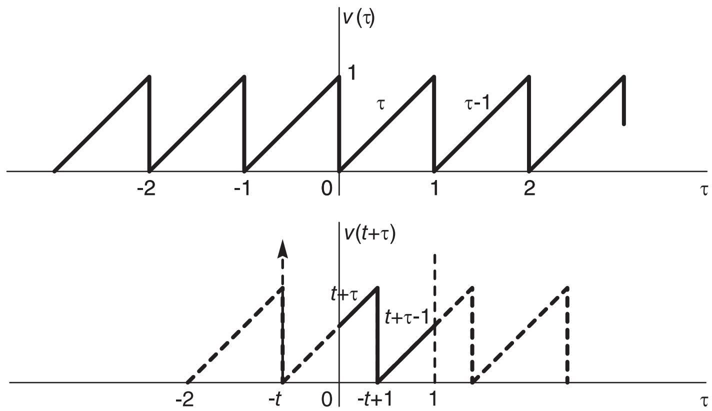

FIGURE 12.18
Periodic ramp and its shifting in time.

$$\begin{aligned} r_{vv}(t) &= \int_0^{1-t} (t+\tau)\,\tau\; d\tau + \int_{1-t}^{1} (t+\tau-1)\,\tau\; d\tau, \quad 0 < t < 1 \\ &= (1/6)\left(2 - 3t + 3t^2\right), \quad 0 < t < 1. \end{aligned}$$

A Fourier series expansion of $r_{vv}(t)$ as a verification produces the trigonometric coefficients

$$a_n = 2\int_0^1 (1/6)\left(2 - 3t + 3t^2\right) \cos n2\pi t\; dt = \frac{1}{2\pi^2 n^2}, \quad n \geq 1$$

and $a_0 = 1/2$ as expected. The functions $S_{vv}(\omega)$ and $r_{vv}(t)$ are shown in Fig. 12.19 and Fig. 12.20, respectively.

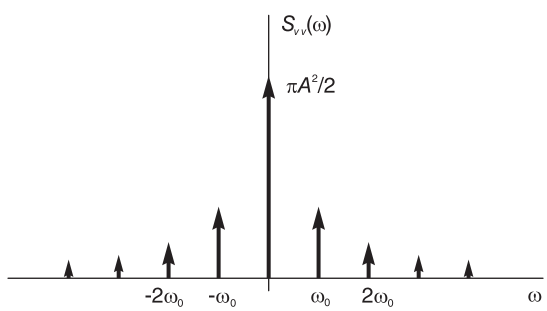

FIGURE 12.19
Power spectral density.

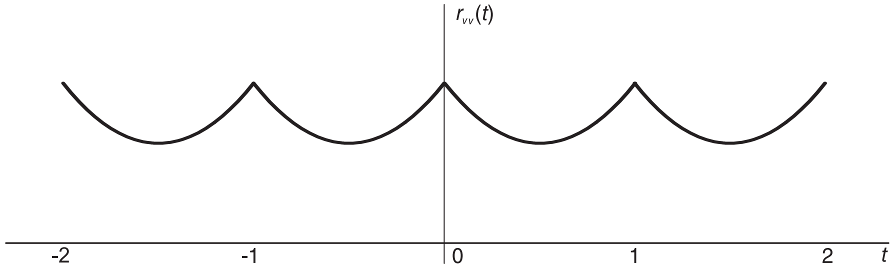

FIGURE 12.20
Autocorrelation of a periodic function.

Example 12.11 *Let*

$$v(t) = A\cos(m\omega_0 t + \theta), \quad m \text{ integer}$$

where $\omega_0 = 2\pi/T$. *Evaluate* $S_{vv}(\omega)$ *and* $r_{vv}(t)$.

We have

$$V_n = \begin{cases} (A/2)\, e^{j\theta}, & n = \pm m \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} S_{vv}(\omega) &= 2\pi \left\{ |V_m|^2 \delta(\omega - m\omega_0) + |V_{-m}|^2 \delta(\omega + m\omega_0) \right\} \\ &= \frac{\pi A^2}{2} \left\{ \delta(\omega - m\omega_0) + \delta(\omega + m\omega_0) \right\} \end{aligned}$$

$$r_{vv}(t) = 2\left\{ (A^2/4)\cos m\omega_0 t \right\} = (A^2/2)\cos m\omega_0 t.$$

12.14 Average, Energy and Power of a Sequence

As noted in Chapter 1 the average value of a sequence $x[n]$ is

$$\overline{x[n]} = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]. \tag{12.134}$$

A real sequence $x[n]$ is an energy sequence if it has a finite energy which can be defined as

$$E = \sum_{n=-\infty}^{\infty} x[n]^2. \tag{12.135}$$

A real aperiodic sequence $x[n]$ is a power sequence if it has a finite average power

$$P = \overline{x[n]^2} = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]^2. \tag{12.136}$$

If the sequence is periodic of period N its average power would be

$$P = \overline{x[n]^2} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]^2. \tag{12.137}$$

Example 12.12 *Let the sequence* $x[n] = 3^{-n}\, u[n]$. *Evaluate its energy.*

$$E = \sum_{n=0}^{\infty} 3^{-2n} u[n] = \sum_{n=0}^{\infty} 9^{-n} u[n] = \frac{1}{1-9^{-1}} = \frac{9}{8}.$$

Example 12.13 *Evaluate the power of the signal*

$$x[n] = 10\cos(\pi n/8).$$

The period N *is deduced from*

$$x[n+N] = x[n]$$

$$10\cos(\pi n/8) = 10\cos[\pi(n+N)/8] = 10\cos(\pi n/8 + \pi N/8)$$

N *is the least value satisfying*

$$(\pi/8)\, N = 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$

$$N = 16$$

$$\bar{P} = \frac{1}{16}\left\{100\sum_{n=0}^{15}\cos^2(\pi n/8)\right\} = \frac{100}{16}\times 2\sum_{n=0}^{7}\cos^2(\pi n/8)$$
$$= \frac{25}{2}(1 + 0.8536 + 0.5 + 0.1464 + 0 + 0.1464 + 0.5 + 0.8536) = 50.$$

12.15 Energy Spectral Density of a Sequence

The energy of a sequence $x[n]$is given by

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

The energy spectral density is given by $\varepsilon_x(\Omega) = \left|X\left(e^{j\Omega}\right)\right|^2$.

Pareseval's Relation states that

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X\left(e^{j\Omega}\right)\right|^2 d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon(\Omega) d\Omega.$$

12.16 Autocorrelation of an Energy Sequence

The autocorrelation of a real energy sequence is given by

$$r_{xx}[n] = \sum_{n=-\infty}^{\infty} x[n+m]\, x[m] = x[n] * x[-n].$$

Its Fourier transform is

$$R_{xx}\left(e^{j\Omega}\right) = X\left(e^{j\Omega}\right) X^*\left(e^{j\Omega}\right) = \left|X\left(e^{j\Omega}\right)\right|^2.$$

12.17 Power density of a Sequence

The power of a sequence is given by

$$P = \overline{x^2[n]} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2.$$

The autocorrelation of a power sequence $x[n]$ is given by

$$r_{xx}[n] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x[n+k]x[k]$$

The power spectral density is given by

$$S_x(\Omega) = \mathcal{F}[r_{xx}[n]] = R_{xx}\left(e^{j\Omega}\right)$$

Parseval's relation takes the form

$$P = \overline{x^2[n]} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\Omega) d\Omega$$

12.18 Passage through a Linear System

Let $x[n]$ be the input and $y[n]$ the output of a linear time-invariant discrete-time system.

If $x[n]$ is an energy sequence its energy spectral density is $\varepsilon_x(\Omega) = \left|X\left(e^{j\Omega}\right)\right|^2$ and that of the output is

$$\varepsilon_y(\Omega) = \left|Y\left(e^{j\Omega}\right)\right|^2 = \left|X\left(e^{j\Omega}\right)\right|^2 \left|H\left(e^{j\Omega}\right)\right|^2.$$

If $x[n]$ is a [power sequence its energy spectral density is $S_x(\Omega)$ and that of the output is

$$S_y(\Omega) = S_x(\Omega) \left|H\left(e^{j\Omega}\right)\right|^2.$$

12.19 Problems

Problem 12.1 *A system has the impulse response*

$$h(t) = \sin \pi t \Pi_T(t) = \sin \pi t \{u(t) - u(t-T)\}.$$

The system receives the ideal impulse Train $\rho_T(t)$ as input

$$x(t) = \rho_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

a) Evaluate the output $y(t)$ of the system if

i) $T = 11$ sec,

ii) $T = 12$ sec.

Evaluate its Fourier Transform $Y(j\omega)$ and its Fourier series expansion with analysis interval T.

b) With $T = 12$ sec evaluate the energy and power spectral densities of $h(t)$ and $y(t)$. Write the expressions describing the auto-correlation of $h(t)$ and $y(t)$ in terms of their spectral densities.

Problem 12.2 *A signal $f(t)$ has a Fourier Transform*

$$F(j\omega) = 14\pi\delta(\omega) + j6\pi\delta\left(\omega - 2\pi\times 10^3\right) - j6\pi\delta\left(\omega + 2\pi\times 10^3\right) + 2\pi\delta\left(\omega - 8\pi\times 10^3\right) + 2\pi\delta\left(\omega + 8\pi\times 10^3\right).$$

a) Is the signal $f(t)$ an energy or power signal?
b) Evaluate the spectral density of $f(t)$.
c) What is the average power of $f(t)$?
d) What is the energy of the signal over an interval of 10^{-3} sec?
e) The signal $f(t)$ is filtered by an ideal bandpass filter with a pass-band $1000\pi < |\omega| < 6000\pi$ r/s and gain K. Evaluate the filter output $g(t)$. What is the average power of $g(t)$?

Problem 12.3 *Let*

$$x(t) = f(t) + g(t)$$

where

$$f(t) = A_1 \sin(\omega_1 t + \theta_1)$$
$$g(t) = A_2 \sin(\omega_2 t + \theta_2)$$

where $\omega_2 > \omega_1$.
a) Evaluate $S_x(\omega)$ the power spectral density of $x(t)$.
b) What is the average power of the component of $x(t)$ of frequency ω_2? A signal $y(t)$ is generated as

$$y(t) = f(t)\, g(t).$$

c) Evaluate the power spectral density $S_y(\omega)$.
d) The signal $y(t)$ is fed to a filter of frequency response

$$H(j\omega) = K\ \Pi_{\omega_2}(\omega).$$

Evaluate the power spectral density at the filter output $z(t)$.

Problem 12.4 *a) Evaluate the function $f(t)$ that is the inverse Laplace transform of the function*

$$F(s) = \left\{1 - e^{-(s+1)}\right\} / (s+1).$$

b) Evaluate the autocorrelation $r_{ff}(t)$ of the function $f(t)$ and its Fourier transform $R_{ff}(j\omega)$.
c) Can the Fourier transform $F(j\omega)$ of $f(t)$ be evaluated from $F(s)$ by letting $s = j\omega$? Justify your answer.
d) Evaluate $|F(j\omega)|^2$ and compare it with $R_{ff}(j\omega)$. e) Is $f(t)$ a power or energy signal ? Evaluate the energy or power spectral density of $f(t)$. Evaluate the energy / power of $f(t)$.
f) Let $H(s) = F(s)$ be the transfer function of a linear system. Let the input to the system be the signal

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n).$$

Evaluate the power spectral density of the system response $y(t)$. Evaluate the average power of $y(t)$ in the frequency band $0 < f < 1.5$ Hz.

Problem 12.5 *Consider a signal $x(t)$ of which the auto correlation function is given by*

$$r_{xx}(t) = e^{-|t|}, \quad -\infty < t < \infty.$$

a) Evaluate $\varepsilon_{xx}(\omega)$ the energy spectral density of $x(t)$.
b) Evaluate the total energy of $x(t)$.
c) The signal $x(t)$ is fed as the input of a filter of frequency response

$$H(j\omega) = \begin{cases} A, & 2 < |\omega| < 4 \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the total energy of the signal $y(t)$ at the filter output.

Problem 12.6

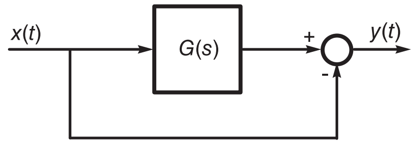

FIGURE 12.21
System block diagram.

In the system shown in Fig. 12.21 the transfer function $G(s)$ *is that of a causal system and is given by*

$$G(s) = 100\pi/(s + 100\pi).$$

a) Evaluate the system impulse response between the input $x(t)$ *and the output* $y(t)$
b) Given that the input is

$$x(t) = 1 + \cos 120\pi t$$

evaluate the average normalized power of the output $y(t)$*. Evaluate the power spectral density of* $y(t)$*.*

Problem 12.7 *Consider the signals*

$$x(t) = \sum_{n=-\infty}^{\infty} \{u(t - 2n) - u(t - 1 - 2n)\}$$

$$y(t) = e^{-t} u(t)$$

which represent voltage potentials in Volt as functions of time t in seconds.
a) For each of the two signals evaluate the total normalized energy and the average normalized power.
b) The signals $z(t)$ *and* $v(t)$ *are given by* $z(t) = x(t)\, y(t)$ *and* $v(t) = x(t) * y(t)$*. For each of these signals state whether the signal is an energy or power signal, explaining why.*

Problem 12.8 *The frequency transformation*

$$s \to (s^2 + 1)/s$$

is applied to a second order lowpass Butterworth filter prototype.
a) Write down the transfer functions $H_{LP}(s)$ *and* $H_{BP}(s)$ *of the lowpass and bandpass filters.*
b) Evaluate the central frequency ω_0 *and the low and high edge frequencies* ω_L *and* ω_H *of the bandpass filter.*
c) Re-write the values of $H_{LP}(s)$ *and* $H_{BP}(s)$ *so that the filter maximal gain be 14 dB. Let the input to this bandpass filter be* $x(t) = 10 + 7\sin\omega_0 t$*. Evaluate the average normalized power of the output* $y(t)$*.*

Problem 12.9 *For each of the following signals, which are expressed in Volt as function of time in seconds, state whether it is an energy or power signal and evaluate its total normalized energy or average normalized power.*
a)
$v(t) = 3\sin[1000\pi(t + 0.0025)] + 2\cos(1500\pi t + \pi/5)$.
b) $w(t) = \begin{cases} 0.25(t-2), & 2 < t < 6 \\ 0, & \text{otherwise.} \end{cases}$
c)
$$x(t) = \sum_{n=0}^{10} w(t - 10n).$$
d)
$$y(t) = \sum_{n=-\infty}^{\infty} w(t - 5n).$$

Problem 12.10 *Let $x(t)$ be a function, $X(j\omega)$ its Fourier transform and*

$$|X(j\omega)| = 1/\sqrt{1+\omega^2} + \pi/2\,\{\delta(\omega-\beta) + \delta(\omega+\beta)\}.$$

a) What is the average value of $x(t)$?
b) Is $x(t)$ periodic? If yes what is its period? If not why?
c) The signal $x(t)$ is applied as the input to a filter of frequency response $H(j\omega)$, where

$$|H(j\omega)| = \Pi_{2\beta}(\omega), \quad \arg[H(j\omega)] = -\pi\omega/(4\beta).$$

Sketch the amplitude spectrum $|Y(j\omega)|$ of the filter output $y(t)$.
d) Let $z(t) = x(t) + 0.5\sin(2.5\beta t) + 0.5$. Sketch the amplitude spectrum $|Z(j\omega)|$ of the signal $z(t)$.

Problem 12.11 *For each of the following signals evaluate the signal total energy and the average normalized power and deduce whether it is an energy or power signal:*
a) $v(t) = A\sin(2000\pi + \pi/3)$.
b) $w(t) = A\sin(2000\pi + \pi/3)\,R_{0.001}(t)$, where

$$R_{0.001}(t) = u(t) - u(t-0.001).$$

c) $x(t) = \displaystyle\sum_{n=-\infty}^{\infty} e^{-(t-5n)}\{u(t-5n) - u(t-5-5n)\}.$
d) $z(t) = A$.

Problem 12.12 *A system of transfer function*

$$H(s) = \left.\frac{K}{s+1}\right|_{s\,\rightarrow\,s/\omega_c}$$

receives an input $x(t)$ and produces an output $y(t)$. Assuming $x(t) = A\cos\omega_0 t$, where $A = 5$ Volt and $\omega_0 = 2\pi f_0 = 2\pi\times 500$ Hz.
a) With $K = 1$ and $\omega_c = 500\pi$ r/s, evaluate the average power of the signal $y(t)$.
b) With $K = 1$ find the value of ω_c so that the average power of $y(t)$ be 5 Watt.
c) With $\omega_c = 1000\pi$ r/s evaluate K so that the average power of $y(t)$ be 5 Watt.

Problem 12.13 *Given the signals $v(t) = x(t)\,y(t)$ and $f(t) = x(t) * z(t)$, where*

$$x(t) = 5R_3(t) = 5[u(t) - u(t-3)]$$

$$y(t) = 2\Pi_{0.5}(t) = 2[u(t+0.5) - u(t-0.5)]$$

$$z(t) = 1 + \cos(\pi t + \pi/3).$$

a) Evaluate $V(j\omega)$ and $F(j\omega)$, the Fourier transforms of $v(t)$ and $f(t)$ as well as the Fourier series coefficients F_n of $f(t)$.
b) State whether each of the signals $v(t)$ and $f(t)$ is an energy or power signal, evaluating the energy or power spectral density, the total energy or the average normalized power in each case.

Problem 12.14 *A signal $f(t)$ of average value $\overline{f(t)} = 15$ is applied to the input of a linear system of impulse response*

$$h(t) = 5e^{-7t}\sin 5\pi t\; u(t).$$

What is the average value $\overline{y(t)}$ of the system output $y(t)$?

Problem 12.15 *A signal $x(t)$ has a Fourier transform*

$$X(j\omega) = 2\pi\; Sa(\omega/400)\, e^{-j\omega/100} \sum_{n=-\infty}^{\infty} \delta(\omega - 100\pi n).$$

The signal is applied to the input of a filter of frequency response $H(j\omega)$ *and output* $y(t)$*, where*

$$|H(j\omega)| = \begin{cases} 1 - [(\omega - 300\pi)/(200\pi)]^2, & 100\pi < |\omega| < 500\pi \\ 0, & \text{otherwise} \end{cases}$$

$$\arg[H(j\omega)] = \begin{cases} -\pi/2, & \omega > 0 \\ \pi/2, & \omega < 0. \end{cases}$$

a) Evaluate the exponential Fourier series coefficients X_n *of* $x(t)$ *with an analysis interval of* 0.02 *sec.*
b) Sketch the frequency response $|H(j\omega)|$.
c) Evaluate the Fourier series coefficients Y_n *of the output* $y(t)$ *over the same analysis period.*
d) Evaluate the output $y(t)$ *and the normalized average power of each components of* $y(t)$.

Problem 12.16 *A system receives an input* $x(t)$ *and produces an output* $y(t)$ *that is the sum of* $x(t)$ *and a delayed version* $x(t-\tau)$ *where* $\tau = 0.4 \times 10^{-3}$ *sec. The signal* $x(t)$ *is a sinusoid of amplitude* 5 *Volt and frequency* 1 *kHz.*
a) Draw the block diagram describing the system.
b) Evaluate the impulse response $h(t)$ *and frequency response* $H(j\omega)$ *of the system between its input* $x(t)$ *and output* $y(t)$.
c) Evaluate and sketch the power spectral density $S_x(\omega)$ *of the signal* $x(t)$*, expressed in terms of the Fourier series coefficients* X_n *of* $x(t)$.
d) Evaluate and sketch the power spectral density $S_y(\omega)$ *and the average power* $\overline{y_2(t)}$ *of the output* $y(t)$.

Problem 12.17 *The signal* $x(t) = e^{-7t}u(t)$ *is applied to the input of a filter of frequency response* $H(j\omega)$ *given by*

$$H(j\omega) = \begin{cases} 5, & 1.1 \leqslant |\omega| \leqslant 3 \\ 0, & \text{otherwise} \end{cases}$$

Evaluate the energy spectral density $\varepsilon_x(\omega)$ *of* $x(t)$ *and* $\varepsilon_y(\omega)$ *of* $y(t)$.

Problem 12.18 *A filter of frequency response*

$$H(j\omega) = \left(1 - \omega^2/W^2\right)\Pi_W(\omega)$$

receives an input $v(t)$ *and produces an output* $y(t)$.

Assuming that the input $v(t)$ *has an autocorrelation* $r_{vv}(t) = \cos(Wt/4)$ *evaluate the power spectral densities* $S_{vv}(\omega)$ *and* $S_{yy}(\omega)$ *of the signals* $v(t)$ *and* $y(t)$*, respectively. Evaluate the normalized average power of* $y(t)$.

Problem 12.19 *Consider the signal*

$$v(t) = 10\sin\beta t \Pi_{T/2}(t)$$

where $\beta = 4\pi/T$.
a) Sketch the signal $v(t)$*. Evaluate its energy and normalized average power and corresponding spectral density if any.*
b) What is the result of integrating the evaluated spectral density?

Problem 12.20 *A signal is given by*

$$v(t) = 10\cos[\beta(t-1)] + 5\sin[4\beta(t-2)] + 8\cos[10\beta(t-3)]$$

where $\beta = 2\pi/T$ *and* $T = 1$ *sec.*
a) Evaluate the exponential Fourier series coefficients of $v(t)$ *with an analysis interval of one second.*
b) Evaluate the signal power spectrum.

TABLE 12.1
Amplitude and phase spectra.

Frequency kHz	0	10	20	30	40	≥ 50
$\lvert F_n \rvert$ Volt	2	2.5	3.5	2	1	0
$\arg[F_n]$ deg.	0	-10	-20	-30	-40	–

Problem 12.21 *A spectrum analyzer displays the amplitude spectrum in Volt and phase spectrum in degrees as the Fourier series coefficients F_n versus the frequency in Hz of a function $f(t)$ as shown in Table 12.1 and with $F_{-n} = F_n^*$.*
a) What is the period τ and the average value of the function $f(t)$?
b) Write the value of the function $f(t)$ as a sum of real expressions.
c) The signal $f(t)$ is fed to a filter of frequency response $H(j\omega)$ where

$$|H(j\omega)| = \Pi_B(\omega)$$

where $B = 50000\pi$ rad/sec, $\arg[H(j\omega)] = -(10^{-3}/180)\omega$ rad/sec and the filter output $g(t)$ is modulated by the carrier $\cos(40000\pi t)$ producing an output $y(t)$. Sketch the Fourier transforms $G(j\omega)$ and $Y(j\omega)$ of $g(t)$ and $y(t)$.
d) What is the average power of the output signal $y(t)$?

Problem 12.22 *Consider the signal:*

$$v(t) = u(t + t_0) - u(t - b + t_0)$$

where $b > t_0 > 0$.
a) Evaluate the autocorrelation $r_{vv}(t)$ of $v(t)$.
b) Evaluate the Fourier transform $R_{vv}(j\omega)$ of $r_{vv}(t)$.
c) Evaluate the Fourier transform $V(j\omega)$, the energy spectral density and deduce therefrom the total energy of $v(t)$. Compare the result with $R_{vv}(j\omega)$.

Problem 12.23 *Evaluate the energy spectral density for each of the following signals:*
a) $x(t) = e^t [u(t) - u(t-1)]$.
b) $y(t) = e^{-t} \sin(t) u(t)$.

Problem 12.24 *Given the signal* $v(t) = e^{-t} u(t)$
a) evaluate the energy of the signal $v(t)$,
b) evaluate the energy of the signal contained in the frequency range 0 to 1 Hz.

Problem 12.25 *Given the signal* $v(t) = e^{-t} u(t)$.
a) Show that $v(t)$ is an energy signal.
b) Evaluate the energy spectral density of $v(t)$.
c) Evaluate the normalized energy contained in the frequency range 0 to 1 r/s.
d) Evaluate the normalized energy contained in the frequency range 0 to 1 Hz.
e) Evaluate the auto-correlation function $r_{vv}(t)$ of $v(t)$.
f) Show how from $r_{vv}(t)$ you can deduce the energy spectral density of $v(t)$.

Problem 12.26 *The signal* $v(t) = 4e^{-2t} u(t)$ *is applied to the input of a filter of frequency response $H(j\omega)$.*
a) What is the total normalized energy E_v of $v(t)$?
b) What is the total normalized energy E_y of the signal $y(t)$ at the filter output in the case where the filter is an ideal lowpass filter of unit gain and cut-off frequency 2 r/s?
c) What is the total normalized energy E_y of the signal $y(t)$ at the filter output in the case where the filter is an ideal bandpass filter of unit gain and pass band extending from 1 to 2 Hz?
d) What is the total normalized energy E_y of the signal $y(t)$ at the filter output in the case where the filter transfer function is $H(s) = 1/(s+2)$?
e) What is the total normalized energy E_y of the signal $y(t)$ at the filter output in the case where the filter frequency response is $H(j\omega) = e^{-j\omega T}$, where T is a constant?

Problem 12.27 *Each of the following signals is given in Volt as a function of the time t in seconds. For each signal evaluate the total energy if it is an energy signal or the average power if it is a power signal.*
a) $x_a(t) = 3[u(t - T_a) - u(t - 6T_a)]$, *where* $T_a > 0$.
b) $x_b(t) = x_a(t)\cos(2\pi t/T_b)$, *where* $T_b = T_a$.
c) $x_c(t) = \sum_{n=-\infty}^{+\infty} x_b(t - nT_c)$, *where* $T_c = 15T_a$.
d) $x_d(t) = x_a(t) + 1$.

Problem 12.28 *Consider the three signals* $x(t)$, $y(t)$ *and* $z(t)$:

$$x(t) = u(t) - u(t-1), \quad y(t) = u(t+0.5) - u(t-0.5), \quad z(t) = \sin(\pi t).$$

a) Is the sum $v(t) = x(t) + y(t)$ *an energy or power signal? Depending on the signal type, evaluate the total normalised energy or the average normalized power, respectively.*
b) Is the convolution $s(t) = x(t) * z(t)$ *an energy or power signal? Depending on the signal type, evaluate the energy spectral density or the power spectral density, respectively.*

Problem 12.29 *Evaluate the power spectral density and the average power of the following periodic signals:*
a) $v(t) = 5\cos(2000\pi t) + 3\sin(500\pi t)$.
b) $x(t) = [1 + \sin(100\pi t)]\cos(2000\pi t)$.
c) $y(t) = 4\sin^2(200\pi t)\cos(2000\pi t)$.
d) $z(t) = \sum_{n=-\infty}^{+\infty} 10^4 \left(t - 10^{-3}n\right)\left\{u\left(t - 10^{-3}n\right) - u\left(t - 10^{-3}[n+1]\right)\right\}$.

Problem 12.30 *Let* $x(t)$ *be a periodic signal having a period* 5×10^{-2} *seconds. Its exponential Fourier series expansion with an analysis interval equal to its period has the Fourier series coefficient*

$$X_n = \begin{cases} 1, & n = 0, \pm 4 \\ \pm j, & n = \pm 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $y(t)$, *be a signal having the Fourier transform* $Y(j\omega) = 150/(125 + j\omega)$.
a) Let $z(t)$ *be the convolution* $z(t) = x(t) * y(t)$. *Evaluate the average power* $\overline{z^2(t)}$ *of* $z(t)$.
b) Let $v(t) = x(t) + y(t)$. *Evaluate the average power* $\overline{v^2(t)}$ *of* $v(t)$.

Problem 12.31 *Let* $x(t) = 3\cos(\omega_1 t) + 4\sin(\omega_2 t)$, *where* $\omega_1 = 120\pi$ *and* $\omega_2 = 180\pi$. *The signal* $x(t)$ *is applied to the input of a filter of transfer function* $H(s) = 1/(1 + 120\pi/s)$.
Evaluate the power spectra density $S_y(\omega)$ *of the the signal* $y(t)$ *at the filter output. Evaluate the average power* $\overline{y^2(t)}$ *of* $y(t)$.

Problem 12.32 *A filter which has a transfer function* $H(s) = K/(1 + s/\omega_c)$ *receives an input signal* $x(t) = A\cos(2\pi f_0 t)$, *where* $A = 5$ *Volt and* $f_0 = 500$ *Hz, and produces an output signal* $y(t)$.
a) Let $K = 1$ *and* $\omega_c = 500\pi$ *r/s. Evaluate the average signal power at the filter output.*
b) Let $K = 1$. *Determine the value of* ω_c *so that the average power of the output signal* $y(t)$ *be 5 Watt.*
c) Let $\omega_c = 1000\pi$ *r/s. Determine the value of* K *so that the average power of the output signal* $y(t)$ *be 5 Watt.*

Problem 12.33 *The periodic signal* $v(t) = \sum_{n=-\infty}^{\infty} (-1)^n \Lambda_{T/4}(t - nT/2)$ *is applied to the input of filter of frequency response* $H(j\omega) = 4\Lambda_{12}(\omega)$ *and output* $y(t)$. *Evaluate*
a) the average power of the signal $v(t)$,
b) the average power of $y(t)$ *if* $T = 2\pi/3$,
c) the average power of $y(t)$ *if* $T = \pi/6$.

Problem 12.34 *A voltage* $v_E(t)$ *is applied to the input of a first order lowpass RC filter with* $RC = 1$*, of which the output is* $v_S(t)$*. For each of the following cases evaluate the average power of the input and output signal* $v_E(t)$ *and* $v_S(t)$*, respectively.*
a) The power spectral density of $v_E(t)$ *is* $S_{v_E}(\omega) = A[\delta(\omega+1) + \delta(\omega-1)]$.
b) The power spectral density of $v_E(t)$ *is* $S_{v_E}(\omega) = u(\omega+1) - u(\omega-1)$.
c) The power spectral density of $v_E(t)$ *is* $S_{v_E}(\omega) = A$.

Problem 12.35 *The signal* $x(t) = \sin(4\pi t)$ *is applied to the input of a filter of transfer function* $H(s) = 1/(s+1)$ *and output* $y(t)$.
a) Evaluate the power spectral density $S_x(\omega)$ *of the signal* $x(t)$.
b) Evaluate the average power of the signal $x(t)$.
c) Evaluate the normalized energy of one period of the signal $x(t)$.
d) Evaluate the power spectral density $S_y(\omega)$ *of the signal* $y(t)$ *at the filter output.*
e) Evaluate the average power $\overline{y^2(t)}$ *of the filter output signal* $y(t)$.

Problem 12.36 *The signal* $v(t) = \sum_{n=-\infty}^{\infty} \delta(t-12n)$ *is applied to the input of a linear system of impulse response* $h(t) = \sin(\pi t)[u(t) - u(t-12)]$*. Evaluate the power spectral density of the filter output signal* $y(t)$.

Problem 12.37 *Let* $x(t)$ *be a periodic signal of period* 5×10^{-3} *seconds and exponential Fourier series coefficients* X_n*, evaluated with an analysis interval equal to its period, given by*

$$X_n = \begin{cases} 1, & n = \pm 1 \\ \pm j/5, & n = \pm 2 \\ (1 \mp 2j)/10, & n = \pm 4 \\ 0, & \text{otherwise.} \end{cases}$$

The properties of the message $m(t)$ *are* $\overline{m(t)} = 0$ *Volt,* $\overline{m^2(t)} = 2$ *Watt,* $|m(t)|_{max} = 5$ *Volt.* $M(f) = 0$ *for* $|f| > 7.5 \times 10^3$ *Hz.*

For each of the five possible frequency responses of the bandpass filter evaluate the maximum amplitude of the modulated signal $y(t)$.

Defining the Harmonic Distortion Rate HDR as

$$HDR = \frac{P_h}{P_T} \times 100\%$$

where P_h *is the average power of the signal harmonics other than the fundamental and* P_T *is the total signal average power.*
a) Evaluate the HDR of the signal $x(t)$.
b) The signal $x(t)$ *is applied to the input of a filter the transfer function of which is given by*

$$H(s) = \left. \frac{1}{s+1} \right|_{s \to s/(400\pi)}.$$

Evaluate the HDR of the filter output signal $y(t)$.

Problem 12.38 *Let* $x(t) = v(t) + a\,v(t-t_0)$*, where* $v(t)$ *is a power signal and* t_0 *is a constant.*
Show that $\overline{x^2(t)} = (1+a^2)\,\overline{v^2(t)} + 2a\,r_v(t_0)$*, where* $\overline{x^2(t)}$ *is the average power of* $x(t)$*,* $\overline{v^2(t)}$ *is that of* $v(t)$ *and* $r_v(t_0)$ *is the autocorrelation function of* $v(t)$ *evaluated at* $t = t_0$.

12.20 Answers to Selected Problems

Problem 12.1 a)

i)
$y(t) = \sum_{n=-\infty}^{\infty} \sin \pi (t - 11n) \{u(t - 11n) - u(t - 11n - 11)\}$

$$\begin{aligned} Y(j\omega) &= 2\pi \sum_{n=-\infty}^{\infty} H_n \delta(\omega - n\omega_0) \\ &= -j\pi \sum_{n=-\infty}^{\infty} \left[e^{-jn\pi + j\beta T/2} Sa(n\pi - \beta T/2) - e^{-jn\pi - j\beta T/2} Sa(n\pi + \beta T/2) \right] \delta(\omega - n\omega_0) \end{aligned}$$

$Y(j\omega) =$
$j\pi \sum_{n=-\infty}^{\infty} \left[e^{-jn\pi + j11\pi/2} Sa(n\pi - 11\pi/2) - e^{-jn\pi - j11\pi/2} Sa(n\pi + 11\pi/2) \right] \delta(\omega - n2\pi/11)$

ii)
$$Y(j\omega) = -j\pi \{\delta(\omega - \pi) - \delta(\omega + 6)\}$$

$$Y_n = \left\{ \begin{array}{ll} \mp j/2\,, & n = \pm 6 \\ 0\,, & n \neq \pm 6 \end{array} \right\}$$
$$b) \quad h(t) = \sin \pi t \{u(t) - u(t - 12)\}$$

b) $h(t) = \sin \pi t \{u(t) - u(t - 12)\}$.

$$\varepsilon_h(t) = (T^2/4) \left| e^{-j(\omega - \pi)T/2}\ Sa\{(\omega - \pi)T/2\} - e^{-j(\omega + \pi)T/2} Sa\{(\omega + \pi)T/2\} \right|^2$$

$$y(t) = \sin \pi t.$$
$$S_y(\omega) = (\pi/2)\{\delta(\omega - \pi) + \delta(\omega + \pi)\}$$

Problem 12.2
a) The signal, having an impulsive spectrum, is periodic. b)

$$\begin{aligned} S_f(\omega) = 98\pi\delta(\omega) + 18\pi \left\{ \delta\left(\omega - 2\pi \times 10^3\right) + \delta\left(\omega + 2\pi \times 10^3\right) \right\} \\ +2\pi \left\{ \delta\left(\omega - 8\pi \times 10^3\right) + \delta\left(\omega + 8\pi \times 10^3\right) \right\} \end{aligned}$$

c)
$$P = \overline{f^2(t)} = \sum_{n=-\infty}^{\infty} |F_n|^2 = 49 + 2 \times 9 + 2 \times 1 = 69$$

d)
$$P = \frac{1}{T}E, \quad E = TP = \frac{2\pi}{\omega_0} \times 69 = 69 \times 10^{-3}$$

e)
$$G_n = \begin{cases} \pm j\,3\,K, & n = \pm 1 \\ 0, & n \neq \pm 1 \end{cases}$$
$$\overline{g^2(t)} = \sum_{n=-\infty}^{\infty} |G_n|^2 = 2 \times 9\ K^2 = 18\ K^2$$

Problem 12.5
a)
$\varepsilon_{xx}(\omega) = \frac{2}{1+\omega^2}$
b) $E = 1$

$$E = 1$$

c) $E = 0.4373 A^2/\pi$
Problem 12.6

$$Y_n = \begin{cases} \frac{\mp j\beta}{2(100\,\pi \pm j\beta)}, & n = \pm 1 \\ 0\,, & otherwise \end{cases}$$

$$S_y(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |Y_n|^2 \delta(\omega - n\omega_0) = 2\pi \times 0.1475 \{\delta(\omega - \beta) + \delta(\omega + \beta)\}$$

$\overline{y^2(t)} = 0.295$

Problem 12.7

See Fig. 12.22

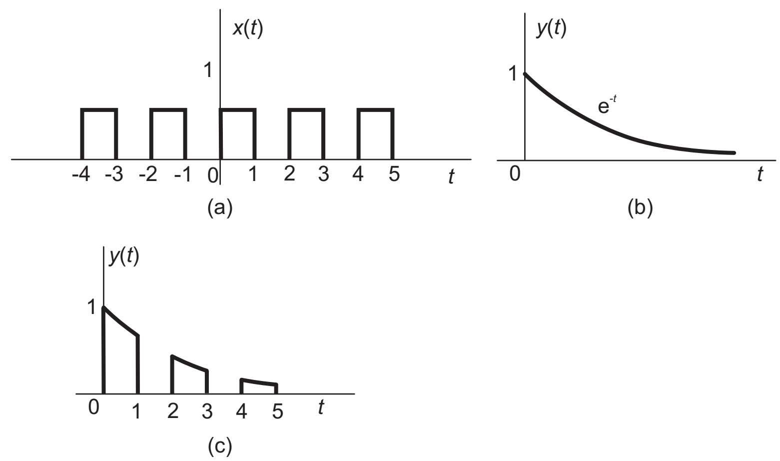

FIGURE 12.22
Figure for Problem 12.7

a) $E_x = \infty E_y(t) = 1/2$ Joules

b) The average normalized powers are $\overline{x^2(t)} = (1/2) \cdot 1 = 1/2$ Watt.
$\overline{y^2(t)} = 0$.

$y(t)$ is an energy signal since $E_y < \infty$, $z(t)$ is periodic since $x(t)$ is periodic. The signal $z(t)$ is therefore a power signal.

Problem 12.8

a)

$$H_{LP}(s) = \frac{K}{s^2 + 1.4142\,s + 1}$$

$$H_{BP}(s) = \frac{K\,s^2}{(s^2+1)^2 + 1.4142\,s\,(s^2+1) + s^2}$$

$K = 1$.

b)

$$\omega_L = 1.6180$$

c)

$$|H_{BP}(j\omega_0)| = 5.01$$

$y(t)$ a sinusoid of amplitude $A = 35.07$, average normalized power 614.95 Watt.

Problem 12.9

a)
$\overline{v^2(t)} = 6.5$ Watt.

b) *Energy signal, being of finite duration
$E_w = \int_0^4 \left(t^2/16\right) dt = 1.333$ joules
c) $E_x = 11\ E_w = 14.63$ joules
d) $\overline{y^2(t)} = \frac{1}{5} E_w = 0.267$ Watt.

Problem 12.10

a) $\overline{x(t)} = 0$ since $X(j\omega)$ has no impulse at the origin $\omega = 0$.
b) $x(t)$ is not periodic. To be periodic the spectrum has to be composed solely of impulses.
c) See Fig.12.23

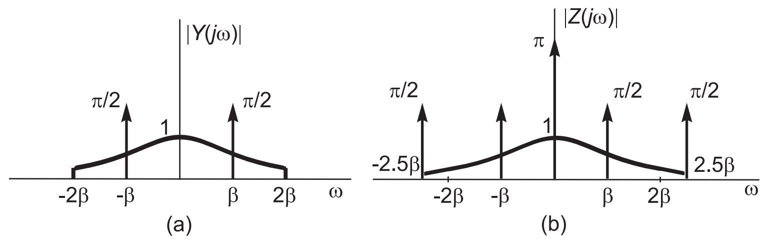

FIGURE 12.23
Figure for Problem 4.10

Problem 12.11

a) Total Energy $= A^2/2$ watt. Power signal
b) Total Energy $= A^2/2000$ Joule. Average normalized power = 0. Energy signal [equal to a single period of $v(t)$].
c) $\overline{x^2(t)} = \frac{1}{6}\left(1 - e^{-6}\right) = 0 \cdot 15$. Power signal. Energy $= \infty$
d) $\overline{z^2(t)} = A^2$, Power signal. Total Energy $= \infty$.

Problem 12.12

a)
$$\overline{y^2(t)} = 2.5 \text{ Watt}$$
b) Note that the average power of a sinusoid of Amplitude A is $A^2/2$
$$\omega_c = 2565.1 \text{ r/s}$$
c)
$$K = 0.8944$$

Problem 12.13

a)
$$V(j\omega) = 5\ \text{Sa}\,(0.25\omega)\, e^{-j0.25\omega}$$
$$F(j\omega) = 30\pi\delta(\omega) - 10e^{-j7\pi/6}\delta(\omega - \pi) + 10e^{j7\pi/6}\delta(\omega + \pi)$$
$$F_n \begin{cases} 15, n = 0 \\ \mp (5/\pi)\, e^{\mp j7\pi/6}, n = \pm 1 \\ 0, \text{otherwise} \end{cases}$$
b)
$$\varepsilon_v(\omega) = |V(j\omega)|^2 = 25\ \text{Sa}^2\,(0.25\omega)$$
$$P = \overline{f^2(t)} = 230.07 \text{ Watt}$$

Problem 12.14

$$\overline{y(t)} = \overline{f(t)}\, H(0) = 15 \frac{25\pi}{7^2 + (5\pi)^2} = 3.984$$

Problem 12.15

a)

$$X_n = 1,\ -0.9,\ 0.636,\ -0.301,\ 0,\ 0.18$$

for $n = 0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5$ respectively, and $X_n = 0$, otherwise.b)

See Fig. 12.24.

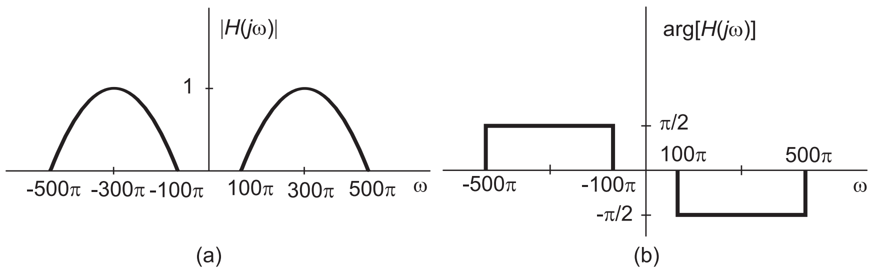

FIGURE 12.24
Amplitude and phase of frequency response, Problem 12.15

c)

$$Y_2 = \mp j0.4775, \quad Y_3 = \pm j0.3001, \quad Y_5 = \mp j0.135, \quad Y_n = 0, \text{ otherwise.}$$

d)

$$y(t) = 0.955 \sin 200\pi t - 0.6 \sin 300\pi t + 0.27 \sin 500\pi t$$

Problem 12.16

a) See Fig. 12.25

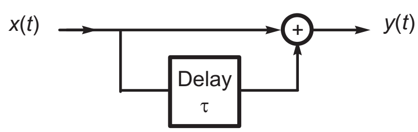

FIGURE 12.25
Figure for Problem 12.16

b)

$$h(t) = \delta(t) + \delta(t-\tau), \quad H(j\omega) = 1 + e^{-j\omega\tau}$$

c)

$$S_x(\omega) = 2\pi \left\{2.5^2 \delta(\omega - 2000\pi) + 2.5^2 \delta(\omega + 2000\pi)\right\}$$

d)

$$S_y(\omega) = 2\pi \times 2.387 \{\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)\}$$

$$\overline{y^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)\, d\omega = 2 \times 2.387 = 4.775 \text{ Watt}$$

See Fig. 12.26

Problem 12.17

a)

$$\varepsilon_x(\omega) = \frac{1}{\omega^2 + 49}$$

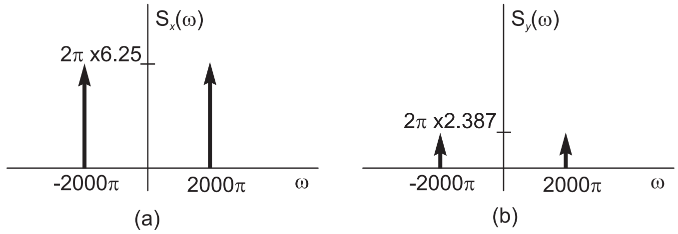

FIGURE 12.26
Figure for Problem 12.16

b)

$$\varepsilon_y(\omega) = \begin{cases} \dfrac{25}{\omega^2+49}, & 1.1 \leqslant \omega \leqslant 1.3 \\ 0, & \text{otherwise} \end{cases}$$

Problem 12.18

$$S_{vv}(\omega) = \pi\left[\delta(\omega - W/4) + \delta(\omega + W/4)\right]$$
$$S_{yy}(\omega) = (15\pi/16)\left[\delta(\omega - W/4) + \delta(\omega + W/4)\right]$$

$$\overline{y^2(t)} = 15/16 = 0.9375 \text{ Watt}$$

Problem 12.19
a) See Fig. 12.27

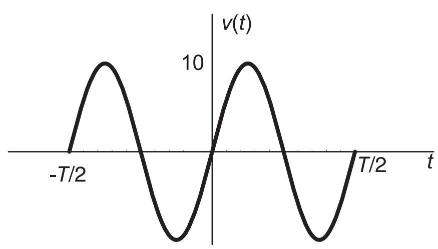

FIGURE 12.27
Figure for Problem 12.19

$$E = 50T, \quad P = 0, \quad \varepsilon_v(\omega) = |V(j\omega)|^2.$$

$$\varepsilon_v(\omega) = 25T^2\left\{Sa^2\left[T(\omega - \beta/2)\right] - 2Sa\left[T(\omega-\beta/2)\right]Sa\left[T(\omega+\beta)/2\right]\right\}$$
$$+25T^2\left\{Sa^2\left[T(\omega+\beta)/2\right]\right\}.$$

b) $100\pi T$

Problem 12.20

$$V_n = \begin{cases} 5, & n = \pm 1 \\ \mp j2.5, & n = \pm 4 \\ 4, & n = \pm 10 \end{cases}$$

$$S_n = |V_n|^2 = \begin{cases} 25, & n = \pm 1 \\ 6.25, & n = \pm 4 \\ 16, & n = \pm 10 \end{cases}$$

Problem 12.22

a)

For $-t_0 \le -t+b-t_0 \le b-t_0$ i.e. $0 \le t \le b$

$$r_{vv}(t) = -t+b-t_0+t_0 = b-t$$

For $-t_0 \le -t-t_0 \le b-t_0$ i.e. $-b \le t \le 0$

$$r_{vv}(t) = b-t_0+t+t_0 = b+t$$

b)

$$R_{vv}(j\omega) = b^2 Sa^2(b\omega/2)$$

c)

$$\varepsilon(\omega) = R_{vv}(j\omega).$$

$$E = b \text{ joules}$$

Problem 12.23

a)

$$|X(j\omega)|^2 = \left(1-2e\cos(\omega)+e^2\right)/\left(1+\omega^2\right)$$

b)

$|Y(j\omega)|^2 = \frac{1}{\omega^4+4}$

Problem 12.24

a) Energy : $\int_0^{+\infty} \left(e^{-t}\right)^2 dt = 0.5.$

b) $V(j\omega)\, 1/\left(1+\omega^2\right)$

Energy= $\frac{1}{2\pi}\int_{-2\pi}^{+2\pi} \frac{1}{1+\omega^2} d\omega = \frac{1}{2\pi}\left[\tan^{-1}(\omega)\right]_{-2\pi}^{+2\pi} = 0.45$

Problem 12.25

a) Energy signal.

b)

The energy spectral density is $1/\left(1+\omega^2\right)$

c)

0.25.

d)

0.45.

e)

$$r_{vv} = 0.5e^{-t}u(t) + 0.5e^{+t}u(-t)$$

f)

$$\mathcal{F}\{r_{vv}(t)\} = 1/\left(1+\omega^2\right)$$

Problem 12.26

a) $E_v = 4.$

b)

$E_y = 2$

c)

$E_y = 0.383.$

d)

$E_y = 0.5$

e)

$E_y = 4.$

Problem 12.27

a)

$E = 45T_a$ Joule.

b)

$E = 22.5T_a$ Joule. $P = 22.5T_a/15T_a = 1.5$ Watt c)

$P = 1$ Watt.

Problem 12.28

a)

$$E = 0.5 + (4 \times 0.5) + 0.5 = 3$$

b)

$$S_s(\omega) = 0.637 \left[\delta(\omega + \pi) + \delta(\omega - \pi)\right]$$

Problem 12.29

5Sol 51

a)

$P = 17$

b) $P = 0.75$

c)

$P = 3$

d)

$P = 33.33$

Problem 12.30

a) $\overline{z^2(t)} = 3.$ b) $\overline{v^2(t)} = 5.$

Problem 12.31

$$S_y(\omega) = 2\pi \times (9/8) \left[\delta(\omega + 120\pi) + \delta(\omega - 120\pi)\right]$$
$$+2\pi \times (36/13) \left[\delta(\omega + 180\pi) + \delta(\omega - 180\pi)\right]$$
$$\overline{y^2(t)} = 7.8$$

Problem 12.32

Sol 54

a) $\overline{x^2(t)} = 2.5.$ b) $\overline{y^2(t)} = 0.4.$ $\omega_c = 2565$ r/s. c) $\overline{y^2(t)} = 0.4.K = 0.894.$